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ON ONE CLASS OF PURITIES

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Abstract: Consider a purity π for the category Λ - mod of all the left Λ -modules, where Λ stands for an associative ring with unit. In this paper there is given a description of the least purity ε_0 with the property $\mathcal{F}_{\varepsilon_0} = \mathcal{F}_\pi$, where $\mathcal{F}_\pi$ denotes the class of all π - flat modules. The results are used for a characterization of rings having only projectively (injectively) closed purities. On the other hand, there are given some examples of purities that are not injectively (projectively) closed.

Key words: Purity, pure flatness, pure divisibility, pure injectivity, pure projectivity, torsion theory.

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1. Consider a purity ω on Λ-*mod* and denote by $\mathcal{F}_\omega$ the class of all ω - flat modules (definitions see below). If ε is a purity, then $\varepsilon \in \mathcal{M}(\omega)$ will mean $\mathcal{F}_\varepsilon = \mathcal{F}_\omega$. We see immediately that there is a purity ε_0 such that $\varepsilon_0 \in \mathcal{M}(\omega)$ and ε_0 is the least with this property. The purpose of this paper is to determine a concrete form of ε_0, provided $\mathcal{F}_\omega$ is closed under submodules and give some applications of the case, when $\mathcal{F}_\omega$ is a torsion - free class (in some torsion theory).

In what follows, by Λ we shall mean a ring with a unity and Λ-*mod* will be the category of left unitary modules over Λ . Let $\mathcal{E}$ be a class of short exact sequences from Λ-*mod* . Denote by $\mathcal{E}_m$ ($\mathcal{E}_e$) the corresponding class of monomorphisms (epimorphisms). The class $\mathcal{E}$ is called a purity if the following conditions are satisfied:

(1) Every split short exact sequence belongs to $\mathcal{E}$.

(2) If α, $\beta \in \mathcal{E}_m$ and $\beta \circ \alpha$ is defined then $\beta \circ \alpha \in \mathcal{E}_m$.

(3) If $\beta \circ \alpha \in \mathcal{E}_m$ and β is a monomorphism then $\alpha \in \mathcal{E}_m$.

(4) If α, $\beta \in \mathcal{E}_e$ and $\beta \circ \alpha$ is defined then $\beta \circ \alpha \in \mathcal{E}_e$.

(5) If $\beta \circ \alpha \in \mathcal{E}_e$ and α is an epimorphism then $\beta \in \mathcal{E}_e$.

If $\mathcal{M}$ is a class of monomorphisms (epimorphisms) then $\mathcal{E}(\mathcal{M})$ will be such a class of short exact sequences that $\mathcal{E}(\mathcal{M})_m = \mathcal{M}$ ($\mathcal{E}(\mathcal{M})_e = \mathcal{M}$) .
Let $\mathfrak{M}$ be a class of modules and let $i(\mathfrak{M})$ $(p(\mathfrak{M}))$ denote the class of all the monomorphisms (epimorphisms) φ such that every module from $\mathfrak{M}$ is injective (projective) with respect to φ . As it is well known, the classes $\mathcal{E}(i(\mathfrak{M}))$ and $\mathcal{E}(p(\mathfrak{M}))$ are purities (see [1] or [2]). Further, if $\mathcal{M}$ is a class of homomorphisms, then $\mathcal{J}(\mathcal{M})$ ($\mathcal{P}(\mathcal{M})$) will be the class of all the modules M such that M is injective (projective) with respect to every morphism from $\mathcal{M}$. If π is a purity, then instead of $\mathcal{J}(\pi_m)$, $\mathcal{P}(\pi_e)$

we shall write $\mathcal{I}_\pi$, $\mathcal{P}_\pi$. A module A is called π-flat (π-divisible) if every short exact sequence with A in the third (first) place belongs to π. The corresponding classes will be denoted by $\mathcal{F}_\pi$ and $\mathcal{D}_\pi$.

2. Throughout this paragraph, let $\mathcal{L}$ denote a non-empty class of modules closed under submodules, isomorphisms and extensions (i.e., if $A, B \in \mathcal{L}$ and $0 \to A \to B \to C \to 0$ is exact then $C \in \mathcal{L}$). Put $k(\mathcal{L}) = \{\varphi / \varphi$ is a monomorphism, $\varphi : A \to B$ and there is a submodule $S \subseteq B$ such that

$$\varphi(A) \cap S = 0 \quad \text{and} \quad B/(\varphi(A) + S) \in \mathcal{L}\}$$

and $\pi = \pi(\mathcal{L}) = \mathcal{E}(k(\mathcal{L}))$. Then $\pi_m = k(\mathcal{L})$.

<u>Theorem 2.1</u>. The class π is a purity.

<u>Proof</u>. (i) Let $\varphi : A \to B$ be a monomorphism and $B = \varphi(A) \oplus C$. Then $\varphi(A) \cap C = 0$ and $B/\varphi(A) + C \in \mathcal{L}$ ($B/\varphi(A) + C = 0$). Thus $\varphi \in k(\mathcal{L}) = \pi_m$.

(ii) Let $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ be two monomorphisms. Without loss of generality we can assume that $A \subseteq B \subseteq C$ and φ, ψ are the canonical monomorphisms.

(α) Let $\varphi, \psi \in \pi_m$. Then there are $S \subseteq B$ and $T \subseteq C$ such that $S \cap A = T \cap B = 0$ and $B/A + S$, $C/B + T \in \mathcal{L}$. Put $X = S + T$. Then $A \cap X = A \cap (S + T) = 0$, as one may check easily. Further $B + T/A + X = B \oplus T/(A + S) \oplus T \cong B/A + S \in \mathcal{L}$ and $C/B + T \in \mathcal{L}$.

Hence the exact sequence

$$0 \longrightarrow {}^{B+T}/_{A+X} \longrightarrow {}^{C}/_{A+X} \longrightarrow {}^{C}/_{B+T} \longrightarrow 0$$

gives ${}^{C}/_{A+X} \in \mathcal{L}$. Thus $\psi \circ \varphi \in \pi_m$.

(β) Let $\psi \circ \varphi \in \pi_m$. There is $T \subseteq C$ such that $A \cap T = 0$ and ${}^{C}/_{A+T} \in \mathcal{L}$. Set $S = B \cap T$. We have $A \cap S = A \cap B \cap T = 0$ and $(A+T) \cap B = A + (T \cap B)$.
Hence ${}^{B}/_{A+S} = {}^{B}/_{A+(B \cap T)} =$
$= {}^{B}/_{(A+T) \cap B} \cong {}^{B+A+T}/_{A+T} \subseteq {}^{C}/_{A+T} \in \mathcal{L}$.

Therefore ${}^{B}/_{A+S} \in \mathcal{L}$ and consequently $\varphi \in \pi_m$.

(iii) Let $A \xrightarrow{\varphi} B \xrightarrow{\psi} C$ be two epimorphisms. Put $X = \mathrm{Ker}\,\varphi$, $Y = \mathrm{Ker}\,\psi$, $Y^{-1} = \{a \mid a \in A,\ \varphi(a) \in Y\}$ (clearly $Y^{-1} = \varphi^{-1}(Y) = \mathrm{Ker}(\psi \circ \varphi)$).
(α) Let $\varphi, \psi \in \pi_\ell$. Hence there are $S \subseteq A$ and $T \subseteq B$ such that $X \cap S = 0 = Y \cap T$ and ${}^{A}/_{X+S}, {}^{B}/_{Y+T} \in \mathcal{L}$.
Since $Y \cap T = 0$, $Y^{-1} \cap T^{-1} = X$ $(T^{-1} = \varphi^{-1}(T))$. If we put $Z = T^{-1} \cap S$, we get $Y^{-1} \cap Z = Y^{-1} \cap T^{-1} \cap S =$
$= X \cap S = 0$. Consider the exact sequence

$$(*) \quad 0 \longrightarrow {}^{Y^{-1}+T^{-1}}/_{Y^{-1}+Z} \longrightarrow {}^{A}/_{Y^{-1}+Z} \longrightarrow {}^{A}/_{Y^{-1}+T^{-1}} \longrightarrow 0 .$$

However

$${}^{Y^{-1}+T^{-1}}/_{Y^{-1}+Z} = {}^{Y^{-1}+Z+T^{-1}}/_{Y^{-1}+Z} \cong {}^{T^{-1}}/_{(Y^{-1}+Z) \cap T^{-1}} =$$

$= T^{-1}/X+Z = T^{-1}/(X+S)\cap T^{-1} \cong T^{-1}+X+S/X+S \subseteq A/X+S \in \mathcal{L}$,

$A/Y^{-1}+T^{-1} \cong A/X\Big/Y^{-1}+T^{-1}/X \cong B/Y+T \in \mathcal{L}$.

Hence from ($*$) we can conclude that $A/Y^{-1}+Z \in \mathcal{L}$ and therefore $\psi \circ \varphi \in \pi_\ell$.

(β) Let $\psi \circ \varphi \in \pi_\ell$. There is $S \subseteq A$ such that $S \cap Y^{-1} = 0$ and $A/S+Y^{-1} \in \mathcal{L}$. From this, $Y \cap$ $\cap\ \varphi(S) = 0$ and

$$B/Y+\varphi(S) \cong A/X\Big/S+Y^{-1}/X \cong A/S+Y^{-1} \in \mathcal{L} .$$

Thus $\psi \in \pi_\ell$.

Theorem 2.2. (i) Let $F \in \mathcal{F}_\pi$. Then there is a submodule $S \subseteq F$ such that $F/S \in \mathcal{L}$ and S is subprojective (i.e. S is isomorphic to a submodule of a projective module).

(ii) Let Λ be left hereditary. Then $F \in \mathcal{F}_\pi$ iff there is a submodule $S \subseteq F$ such that $F/S \in \mathcal{L}$ and S is projective.

(iii) $\mathcal{L} \subseteq \mathcal{F}_\pi$. The equality $\mathcal{L} = \mathcal{F}_\pi$ holds iff $\mathcal{L}$ contains all projective modules from $\Lambda-mod$.

(iv) Let $D \in \Lambda-mod$ and $\hat{D}$ be an injective hull of D . Then $D \in \mathcal{D}_\pi$ iff $\hat{D}/D \in \mathcal{L}$.

(v) Let $P \in \Lambda-mod$. Then $P \in \mathcal{P}_\pi$ iff P is projective with respect to every epimorphism ψ with $\mathcal{I}m\, \psi \in \mathcal{L}$.

(vi) Put $\mathcal{L}^+ = \{A \mid Hom_\Lambda(A,B) = 0 \; \forall B \in \mathcal{L}\}$. Then $\mathcal{L}^+ \subseteq \mathcal{P}_\pi$.

(vii) Let $I \in \Lambda-mod$ Then $I \in \mathcal{J}_\pi$ iff $Ext_\Lambda(B, I) = 0$ for all $B \in \mathcal{L}$.

Proof. (i) Consider an exact sequence $0 \to A \xrightarrow{\alpha} P \xrightarrow{\beta} F \to 0$, where P is projective. Since $F \in \mathcal{F}_\pi$, $\alpha \in \pi_m$. Hence there is $T \subseteq P$ such that $A \cap T = 0$ and $P/A+T \in \mathcal{L}$. Therefore $S = \beta(T) \cong T$ and $F/S \in \mathcal{L}$.

(ii) By (i) and using the fact that every projective module lies in $\mathcal{F}_\pi$ and $\mathcal{F}_\pi$ is closed under extensions.

(iii) If $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ is an exact sequence with $C \in \mathcal{L}$ then $\alpha(A) \cap 0 = 0$ and $B/\alpha(A) \in \mathcal{L}$; so $\alpha \in \pi_m$. On the other hand, if $\mathcal{L}$ contains all projective modules then $\mathcal{F}_\pi \subseteq \mathcal{L}$ by (i).

(iv) If $D \in \mathcal{D}_\pi$ then $\alpha \in \pi_m$; α being the canonical monomorphism of D into $\hat{D}$. But D is essential in $\hat{D}$ and hence $\hat{D}/D \in \mathcal{L}$. Conversely, if $\hat{D}/D \in \mathcal{L}$ then $\alpha \in \pi_m$ and consequently $D \in \mathcal{D}_\pi$ (since $\hat{D} \in \mathcal{D}_\pi$).

(v) Let P satisfy the hypothesis. Let $\beta \in \pi_e$, $\beta : B \to C$ and $\gamma \in Hom_\Lambda(P, C)$ be arbitrary homomorphisms. There is $S \subseteq B$ such that $A \cap S = 0$ and $B/A+S \in \mathcal{L}$, $A = Ker\,\beta$. We can write the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow{\scriptstyle \delta} & & \\ 0 & \longrightarrow & A \oplus S & \xrightarrow{\varphi} & B & \xrightarrow{\psi} & E & \longrightarrow & 0 \end{array}$$

By the hypothesis there is $\mu : P \longrightarrow B$ such that $\psi \circ \mu = \delta \circ \gamma$. Hence $\delta \circ \beta \circ \mu = \psi \circ \mu = \delta \circ \gamma$ and $\mathcal{Im}\ \tau \subseteq \mathcal{Ker}\ \delta$, where $\tau = (\beta \circ \mu) - \gamma$. Further, $\mathcal{Ker}\ \delta = \beta(S)$ and $\sigma = \beta / S$ is an isomorphism of S onto $\mathcal{Ker}\ \delta$. Put $\rho = \sigma^{-1} \circ \tau$, then $\rho : P \longrightarrow B$ and $\beta \circ \rho = \tau$. Thus $\gamma = \beta \circ (\mu - \rho)$, P is projective with respect to β and consequently $P \in \mathcal{P}_{\pi}$.

(vi) By (v).

(vii) Let $\mathrm{Ext}_{\Lambda}(B, I) = 0\ \forall B \in \mathcal{L}$. Consider an π-exact sequence $0 \longrightarrow A \xrightarrow{\alpha} C \xrightarrow{\beta} D \longrightarrow 0$. We show that I is injective with respect to α. For let $\tau : A \longrightarrow I$ be arbitrary. We get the commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & D & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \| & & \downarrow & & \\ 0 & \longrightarrow & \alpha(A) \oplus S & \xrightarrow{\gamma} & C & \longrightarrow & E & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho} & & \downarrow & & \| & & \\ 0 & \longrightarrow & I & \longrightarrow & X & \longrightarrow & E & \longrightarrow & 0 \end{array}$$

where α_1, ρ, γ are defined by obvious manner, $\rho \circ \alpha_1 = \tau$. Since $E \in \mathcal{L}$, the nether row splits and there is $\lambda : C \longrightarrow I$ such that $\lambda \circ \gamma = \rho$. Hence $\tau = \rho \circ \alpha_1 = \lambda \circ \gamma \circ \alpha_1 = \lambda \circ \alpha$.

<u>Theorem 2.3</u>. Let ω be such a purity that $\mathcal{L} \subseteq \mathcal{F}_{\omega}$. Then $\pi \subseteq \omega$.

Proof. Let $\alpha \in \pi_m$, $\alpha : A \longrightarrow B$. There is $S \subseteq$ $\subseteq B$ such that $S \cap \alpha(A) = 0$ and $B/\alpha(A) + S \in \mathcal{L}$. Denote by β the canonical inclusion of A into $\alpha(A) \oplus S$ and by γ that of $\alpha(A) \oplus S$ into B. Then $\alpha =$ $= \gamma \circ \beta$. However, $\gamma, \beta \in \omega_m$ and hence $\alpha \in \omega_m$.

Theorem 2.4. Let Λ be a left hereditary ring and $\mathcal{C}$ be a class of Λ-modules. Then the following conditions are equivalent:

(i) There is a purity σ such that $\mathcal{C} = \mathcal{F}_\sigma$.

(ii) $\mathcal{C}$ is closed under submodules, isomorphisms, extensions and every projective module lies in $\mathcal{C}$.

Proof. (i) implies (ii). This assertion is a well known fact. (ii) implies (i). By 2.2 (iii), taking $\mathcal{C}$ for our class $\mathcal{L}$.

Theorem 2.5. Let ω be a purity and $\mathcal{F}_\omega$ be closed under submodules. Let $\pi(\mathcal{F}_\omega)$ denote the purity corresponding to the class $\mathcal{F}_\omega$ in the sense of 2.1. Then $\pi(\mathcal{F}_\omega) \in$ $\in m(\omega)$ and $\pi(\mathcal{F}_\omega)$ is the least purity with this property.

Proof. By 2.2 and 2.3.

Corollary 2.6. Let Λ be a left hereditary ring and ω be a purity. Then $\pi(\mathcal{F}_\omega) \in m(\omega)$ and $\pi(\mathcal{F}_\omega)$ is the least purity with this property.

Recall that a purity σ is called injectively closed (projectively closed) iff $\sigma = \varepsilon(i(\mathcal{J}_\sigma))$ $(\sigma = \varepsilon(p(\mathcal{P}_\sigma)))$.

Example 2.7. Be p a prime. Consider $\mathcal{M}$ the least class of Abelian groups closed under subgroups, isomorphisms and extensions, containing all cocyclic p-primary groups. Let σ be the purity corresponding to $\mathfrak{M}$ in the sense of 2.1. Put $C = \sum_{i=1}^{\infty} \oplus C_i$, $C_i \cong C(p)$ for all i. According to 2.2 (ii), $C_i \in \mathcal{F}_\sigma$ and $C \notin \mathcal{F}_\sigma$. Hence $\mathcal{F}_\sigma$ is not closed under direct sums and consequently σ cannot be injectively closed (see [3]). Further put $D = \prod_{i=1}^{\infty} C_i$. By 2.2 (iv), $C_i \in \mathcal{D}_\sigma$ and $D \notin \mathcal{D}_\sigma$.

Therefore $\mathcal{D}_\sigma$ is not closed under direct products and henceforth σ is not projectively closed.

Example 2.8. Let Λ be not an S-ring. Hence there is a simple Λ-module M such that $Hom_\Lambda(M,\Lambda) = 0$. Denote by $\mathfrak{M}$ the least class of Λ-modules which is closed under submodules, isomorphisms, extensions and which contains M. Then the corresponding purity is not injectively closed (for the same reason as in the example 2.6).

Theorem 2.9. For a ring Λ the following conditions are equivalent:

(i) Any purity on Λ-mod is injectively closed.

(ii) Λ is semi-simple (artinian).

Proof. (i) implies (ii). Take $\mathfrak{N}$, the least class of modules closed under extensions, isomorphisms, submodules and containing all cyclic modules. Let σ be the corresponding purity. If $I \in \mathcal{J}_\sigma$ then I is injective by 2.2 (vii) and consequently σ_m contains every monomorphism from Λ-mod (since σ is injectively closed). Hence every Λ-module is σ-divisible and so $\hat{M}/M \in \mathfrak{N}\ \forall M \in \Lambda$-$mod$.

However there is a cardinal number α such that $card\, N \leq \alpha \; \forall N \in \mathcal{N}$. Therefore $card\, \hat{M}/M \leq \alpha \; \forall M \in \Lambda\text{-}mod$ and hence Λ is semi-simple.

(ii) implies (i). Obvious.

<u>Theorem 2.10</u>. Let Λ be a commutative ring. Then the following statements are equivalent:

(i) Any purity on $\Lambda\text{-}mod$ is projectively closed.

(ii) The class $\mathcal{D}_\sigma$ is closed under direct products for every purity σ on $\Lambda\text{-}mod$.

(iii) The class $\mathcal{D}_\sigma$ is closed under direct sums for every purity σ on $\Lambda\text{-}mod$.

(iv) Λ is semi-simple artinian.

<u>Proof</u>. (i) implies (ii). See [2].

(ii) implies (iv). First we show that any simple Λ-module is injective (i.e. Λ is a V-ring). For let $M \in \Lambda\text{-}mod$ be simple. Suppose that M is not injective. Then $\hat{M}/M \neq 0$ and $card\, \hat{M}/M = \alpha \geq 2$. Put $\beta = max(\alpha, \aleph_0)$ and denote by $\mathcal{M}$ the least class of modules closed under submodules, isomorphisms, extensions and containing $\hat{M}/M$. If $N \in \mathcal{M}$ then obviously $\beta \geq card\, N$. Let σ be the corresponding purity and S be a set with $card\, S > \beta$. Since M is simple, there is a maximal ideal I in Λ such that $M \cong \Lambda/I$. However Λ is commutative and so $I \cdot M = 0$. Hence $I \cdot D = 0$, where $D = \prod_{s \in S} M_s$, $M_s \cong M \; \forall s \in S$. Therefore $D \cong \sum_{t \in T} \oplus M_t$, $M_t \cong M \; \forall t \in T$.

Obviously $\mathrm{card}\, T \geq \mathrm{card}\, S$. Since D is essential in $X = \sum_{t \in T} \oplus \hat{M}_t$, we have $D \subseteq X \subseteq \hat{D}$. Hence $\mathrm{card}\, \hat{D}/D \geq$ $\geq \mathrm{card}\, X/D \geq \mathrm{card}\, T \geq \mathrm{card}\, S > \beta$ and consequently $\hat{D}/D \notin \mathfrak{M}$. On the other hand, $M \in \mathcal{D}_\sigma$ and hence $\mathcal{D}_\sigma$ is not closed under direct products, a contradiction.

Let now $\mathcal{C} = \{A \mid \mathrm{Hom}_\Lambda(A, N) = 0$ for every simple module $N\}$. Since the simple modules are injective, $\mathcal{C}$ is closed under submodules and consequently $\mathcal{C} = \{0\}$ (since no cyclic module lies in $\mathcal{C}$). Hence every non-zero Λ-module has a proper maximal submodule. Assume that Λ is not semi-simple. Then there is a module A such that $A \neq \hat{A}$ and $\hat{A}/A = M$ is simple (non-zero). Consider $\mathcal{N}$, the least class of modules closed under isomorphisms, extensions, submodules and containing M. However, if $X \in \mathcal{N}$ is non-zero then X is a finite direct sum of copies of M (since M is simple and injective). Let τ be the purity corresponding to $\mathcal{N}$ and B denote $\prod_{i=1}^{\infty} A_i$, $A_i \cong A \ \forall i$.

Since $A \in \mathcal{D}_\tau$ and $\mathcal{D}_\tau$ is closed under direct products, $B \in \mathcal{D}_\tau$. Hence $\hat{B}/B \in \mathcal{N}$ and so $\hat{B}/B \cong M_1 \oplus \dots \oplus M_n$, $M_j \cong M$ for $j = 1, \dots, n$ (the case $\hat{B} = B$ cannot arise, otherwise A should be injective). On the other hand, $B \cong A_1 \oplus \dots \oplus A_{n+1} \oplus C$ and hence $\hat{B} \cong \hat{A} \oplus \dots \oplus \hat{A}_{n+1} \oplus \hat{C}$.

Therefore there is an epimorphism $\psi : \hat{B}/B \longrightarrow Y$, $Y \cong N_1 \oplus \dots \oplus N_{n+1}$; each N_j is isomorphic to M. Since M is simple and Λ commutative, we can consider M

to be a ring and consequently a field. In this case $\dim_M \hat{B}/B = n$ and $\dim_M Y = n+1$, a contradiction.

(iii) implies (iv). Let Λ not be semi-simple. Hence there is $A \in \Lambda\text{-}mod$ such that $card\ \hat{A}/A = \alpha \geq 2$.

Denote by $\mathcal{M}$ the least class of modules closed under isomorphisms, extensions, submodules and containing $\hat{A}/A$.

Then $A \in \mathcal{D}_\varphi$ and $\sum_{i \in L} \oplus A_i \notin \mathcal{D}_\varphi$ where φ is the purity corresponding to $\mathcal{M}$, L is a set with $card\ L > \max(\alpha, \aleph_0)$ and $A_i \cong A\ \forall i \in L$. Thus $\mathcal{D}_\varphi$ is not closed under direct sums, a contradiction.

(iv) implies (iii) and (ii). Obvious.

Theorem 2.11. Let Λ be a left semi-hereditary ring. Then the following conditions are equivalent:

(i) The class $\mathcal{F}_\sigma$ is closed under direct sums for every purity σ on $\Lambda\text{-}mod$.

(ii) Λ is a semi-simple artinian ring.

Proof. By Example 2.8.

3. Let $\mathcal{U}$ be a class of modules closed under quotients, isomorphisms and extensions. Put $\mathcal{L}(\mathcal{U}) = \{\varphi \mid \varphi$ is a monomorphism, $\varphi: A \longrightarrow B$ and there is $S \subseteq B$ such that $S + \varphi(A) = B$ and $S \cap \varphi(A) \in \mathcal{U}\}$ and $\varphi = \varepsilon(\mathcal{L}(\mathcal{U}))$.

Theorem 3.1. (i) The class φ is a purity.

(ii) Let $D \in \mathcal{D}_\varphi$. Then there is a submodule $A \subseteq D$ such that $A \in \mathcal{U}$ and D/A is a homomorphic image of an injective module.

(iii) Let Λ be left hereditary. Then $D \in \mathcal{D}_\varphi$ iff there is a submodule $A \subseteq D$ such that $A \in \mathcal{U}$ and D/A is injective.

(iv) $\mathcal{U} \subseteq \mathcal{D}_\varphi$. The equality $\mathcal{U} = \mathcal{D}_\varphi$ holds iff $\mathcal{U}$ contains all injective modules from $\Lambda\text{-}\mathit{mod}$.

(v) Let $F \in \mathcal{F}_\varphi$. Then there is an exact sequence $0 \longrightarrow A \longrightarrow S \longrightarrow F \longrightarrow 0$ such that $A \in \mathcal{U}$ and S is subprojective.

(vi) Let Λ be left hereditary or left perfect. Then $F \in \mathcal{F}_\varphi$ iff there is an exact sequence $0 \longrightarrow A \longrightarrow P \longrightarrow F \longrightarrow 0$ with P projective and $A \in \mathcal{U}$.

(vii) Let $Q \in \Lambda\text{-}\mathit{mod}$. Then $Q \in \mathcal{I}_\varphi$ iff Q is injective with respect to every monomorphism $\varphi : A \longrightarrow B$ with $A \in \mathcal{U}$.

(viii) Put $\mathcal{U}^* = \{B \mid \mathit{Hom}_\Lambda(A, B) = 0 \ \forall A \in \mathcal{U}\}$. Then $\mathcal{U}^* \subseteq \mathcal{I}_\varphi$.

(ix) Let $P \in \Lambda\text{-}\mathit{mod}$. Then $P \in \mathcal{P}_\varphi$ iff $\mathit{Ext}_\Lambda(P, A) = 0$ for all $A \in \mathcal{U}$.

<u>Proof</u>. Similarly as for 2.1, 2.2.

<u>Theorem 3.2</u>. Let ω be such a purity that $\mathcal{U} \subseteq \mathcal{D}_\omega$. Then $\varphi \subseteq \omega$.

<u>Proof</u>. The proof is dual to that of Theorem 2.3.

<u>Theorem 3.3</u>. Let Λ be a left hereditary ring and $\mathcal{C}$ be

a class of Λ -modules. Then the following conditions are equivalent:

(i) There is a purity σ such that $\mathcal{D}_\sigma = \mathcal{C}$.

(ii) $\mathcal{C}$ is closed under quotients, isomorphisms, extensions and every injective module lies in $\mathcal{C}$.

Proof. By 3.1 (iv).

If ω is a purity then $\varepsilon \in \mathcal{C}(\omega)$ will mean that ε is a purity and $\mathcal{D}_\omega = \mathcal{D}_\varepsilon$.

Theorem 3.4. Let ω be a purity and $\mathcal{D}_\omega$ be closed under quotients. Then $\rho(\mathcal{D}_\omega) \in \mathcal{C}(\omega)$ and $\rho(\mathcal{D}_\omega)$ is the least purity with this property.

4. Let $\mathcal{N}$ be a class of modules closed under quotients and isomorphisms. In [4] there is introduced a special notion of purity, namely the $\mathcal{N}$ -purity, in this way: An exact sequence $0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$ belongs to the $\mathcal{N}$ -purity iff $\alpha(A)$ is a direct summand in every submodule $S \subseteq B$ such that $\alpha(A) \subseteq S \subseteq B$ and $S/\alpha(A) \in \mathcal{N}$. It is an easy exercise to show that the $\mathcal{N}$ -purity is in fact the purity $\varepsilon(\check{p}(\mathcal{N}))$.

Proposition 4.1. Let $\mathcal{N}$ be a class of modules. Put $\tau = \varepsilon(p(\mathcal{N}))$ and $\mathcal{N}^* = \{A \mid Hom_\Lambda(N,A) = 0 \ \forall N \in \mathcal{N}\}$. Then $\mathcal{N}^* \subseteq \mathcal{F}_\tau$. Moreover, if $\Lambda \in \mathcal{N}^*$ and $\mathcal{N}$ is closed under quotients and isomorphisms, then $\mathcal{N}^* = \mathcal{F}_\tau$.

Proof. (i) The inclusion $\mathcal{N}^* \subseteq \mathcal{F}_\tau$ is obvious.

(ii) Let $\mathcal{N}$ satisfy the additional hypotheses. Then every projective module lies in $\mathcal{N}^*$ and τ is the $\mathcal{N}$-purity. For $F \in \mathcal{F}_\tau$ we have an exact sequence $0 \to U \xrightarrow{\alpha} P \xrightarrow{\beta} \to F \to 0$ with $\alpha \in \tau_m$ and P projective. Suppose that $F \notin \mathcal{N}^*$. Hence there is $V \subseteq F$, $V \neq 0$ and $V \in \mathcal{N}$. Since $\alpha \in \tau_m$ and $\beta^{-1}(V)/\alpha(V) \in \mathcal{N}$, $\beta^{-1}(V) \cong \alpha(U) \oplus V$, a contradiction.

Theorem 4.2. Let $(\mathcal{M}, \mathcal{L})$ be a torsion theory and let σ and π denote the $\mathcal{M}$-purity and the purity corresponding to $\mathcal{L}$ in the sense of 2.1 respectively. Then $\sigma = \pi$. Moreover, if $\Lambda \in \mathcal{L}$ then $\mathcal{F}_\sigma = \mathcal{L}$ and σ is the least purity with this property.

Proof. By Proposition 4.1 and Theorem 2.3 we have $\pi \subseteq \sigma$. On the other hand, let $0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$ be a σ-exact sequence and κ be the idempotent radical corresponding to the given torsion theory. Then $\alpha(A) \subseteq T$ and $T/\alpha(A) \cong \kappa(C) \in \mathcal{M}$, where $T = \beta^{-1}(\kappa(C))$.

Hence $T = \alpha(A) \oplus S$. However

$$B/\alpha(A) \oplus S = B/T \cong B/\alpha(A) / T/\alpha(A) \cong C/\kappa(C) \in \mathcal{L}.$$

Thus $\alpha \in \pi_m$ and consequently $\sigma \subseteq \pi$.

For the remaining statements of the theorem - see 4.1 and 2.5.

Theorem 4.3. Let $(\mathcal{M}, \mathcal{L})$ be a torsion theory and σ denote the $\mathcal{M}$-purity. Then

(i) $Q \in \mathcal{J}_\sigma$ iff $\mathrm{Ext}_\Lambda(B, Q) = 0 \ \forall B \in \mathcal{L}$.

(ii) $P \in \mathcal{P}_\sigma$ iff P is a direct summand in a direct sum of a projective module and of a module from $\mathcal{M}$.

(iii) Let Λ be left hereditary. Then $P \in \mathcal{P}_\sigma$ iff P is a direct sum of a projective module and of a module from $\mathcal{M}$.

(iv) $D \in \mathcal{D}_\sigma$ iff $\mathrm{Ext}_\Lambda(M, D) = 0 \ \forall M \in \mathcal{M}$.

(v) $D \in \mathcal{D}_\sigma$ iff $\hat{D}/D \in \mathcal{L}$.

(vi) Let $\hat{\Lambda} \in \mathcal{M}$. Then every module from $\mathcal{D}_\sigma$ is injective.

(vii) If $F \in \mathcal{F}_\sigma$ then $\varkappa(F)$ is subprojective.

(viii) Let Λ be left hereditary. Then $F \in \mathcal{F}_\sigma$ iff $\varkappa(F)$ is projective.

(ix) $\mathcal{L} \subseteq \mathcal{F}_\sigma$. The equality $\mathcal{L} = \mathcal{F}_\sigma$ holds iff $\Lambda \in \mathcal{L}$.

Proof. The statements (i), (ii) and (iii) are proved in [4]. The statement (iv) is a consequence of the fact that σ is projectively closed. The rest by 2.2 and 4.1.

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