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Commentationes Mathematicae Universitatis Carolinae, Vol. 14 (1973), No. 2, 215--222

Persistent URL: <http://dml.cz/dmlcz/105486>

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GENERALIZED MARKUŠEVIČ BASES

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Abstract:

Necessary and sufficient condition for w^* - w -continuity of the Dyer embedding of a B -space X^* with a generalized Markuševič basis to $C_0(H)$ is found.

Key words:

Markuševič basis, weakly compactly generated Banach spaces.

AMS: Primary 46B05

Ref. Ž. 7.972.22

Notations. In this paper, X denotes a real B -space with the norm $\|\cdot\|$, X^* its dual B -space with the w^* -topology w^* , and X^{**} its second dual, \varkappa denotes the natural embedding of X into X^{**} .

For each locally compact topological space H we denote by $C_0(H)$ the B -space of all real continuous functions on H which vanish at infinity with the supremum norm $\|\cdot\|$ and the weak topology w . In the case H is discrete we denote $C_0(H) = c_0(H)$.

For any normed linear space V and any set $M \subset V$ we denote by $\text{lin } M$ the linear span of M in V and

$\text{sp } M$ the norm-closure of $\text{lin } M$.

If Γ is a set, $\Gamma \neq \emptyset$, then $\ell_\infty(\Gamma)$ is the B -space of all bounded mappings from Γ into \mathbb{R} , \mathbb{R} being real numbers.

Definition 1 (see [2]). A B -space X has a generalized Markuševič basis (shortly gMb) (B, γ) , $B \subset X$, $\gamma \subset X^*$ if:

(i) if $b \in B$, $f \in \gamma$, then $b \neq 0$, $f \neq 0$ and $f(b) \in \{0, 1\}$,

(ii) for any finite set $D \subset B$ there exists a finite set $D' \subset B$ so that: $D \subset \text{lin } D'$ and if $f \in \gamma$, then

$$f\left(\sum_{d \in D'} d\right) \in \{0, 1\},$$

(iii) $\text{sp } B = X$,

(iv) γ is total over X and γ is bounded in X^* .

Remark. (a) Each finite set $D \subset B$ such that for any $f \in \gamma$, $f\left(\sum_{d \in D} d\right) \in \{0, 1\}$, is called γ -orthogonal set.

(b) Markuševič basis (shortly Mb) is a gMb for which (B, γ) is a biorthogonal collection.

(c) From the condition (ii) it is easily seen that

$\text{lin } B = \left\{ \sum_{j=0}^m \lambda_j d_j ; \{d_0, \dots, d_m\} \text{ is a } \gamma\text{-orthogonal subset of } B, \{\lambda_0, \dots, \lambda_m\} \subset \mathbb{R}, m \in \omega \right\}$. We will call each such $\sum_{j=0}^m \lambda_j d_j$ (B, γ) -combination.

(d) Evidently any Mb is a gMb. It was proved in [4] that any weakly compactly generated B -space X (i.e. there is a weakly compact $K \subset X$ such that $X = \text{sp } K$) has a Mb. The following two examples show that there is a great diffe-

rence between Mb's and gMb's and the class of all B-spaces with a gMb is very large.

Example 1. Let Γ be a nonempty set. Then $l_\infty(\Gamma)$ has a gMb.

Proof. Let $\gamma = \{\varphi_\alpha; \alpha \in \Gamma\}$ where $\varphi_\alpha(x) = x(\alpha)$ for any $x \in l_\infty(\Gamma)$, $\alpha \in \Gamma$. It is easily seen that for $B = \{x \in l_\infty(\Gamma); x(\alpha) \in \{0, 1\} \text{ for any } \alpha \in \Gamma; x \neq 0\}$ (B, γ) satisfies Conditions (i), (iii), (iv) from Definition 1. We show that (B, γ) satisfies (ii):

Let $\{i_0, \dots, i_n\} \subset B$, $n \in \omega$. For any set $A \subset \{0, \dots, n\}$, we define $d_A \in l_\infty(\Gamma)$ so that $d_A(\alpha) = 1$ iff $A = \{i \in \{0, \dots, n\}; i_i(\alpha) = 1\}$, $d_A(\alpha) = 0$ in all other cases, for any $\alpha \in \Gamma$. Then a finite set $\{d_A; \emptyset \neq A \subset \{0, \dots, n\}\} \subset B$ is γ -orthogonal and for any $i \in \{0, \dots, n\}$

$$i_i = \sum \{d_A; \emptyset \neq A \subset \{0, \dots, n\}, i \in A\}.$$

Example 2. It is easy to see that if (B, γ) is a Mb and $B', B'' \subset B$ so that $B' \cup B'' = B$, $B' \cap B'' = \emptyset$, then $\text{sp } B' \cap \text{sp } B'' = \{0\}$.

The following example shows that in the case of gMb this is far from true:

In the space $c_0(\Gamma)$, Γ a nonempty set, let $\gamma = \{e_\alpha, \alpha \in \Gamma\}$, where $e_\alpha \in l_1(\Gamma)$ so that $e_\alpha(\beta) = 1$ if $\alpha = \beta$, $e_\alpha(\beta) = 0$ if $\alpha \neq \beta$ for any $\alpha, \beta \in \Gamma$. Let $B = \{x \in c_0(\Gamma); x(\alpha) \in \{0, 1\} \text{ for any } \alpha \in \Gamma, x \neq 0\}$. Then (B, γ) is a gMb of $c_0(\Gamma)$. Suppose that $B = B' \cup B''$, $B' \cap B'' = \emptyset$ and $\text{sp } B' \cap \text{sp } B'' = \{0\}$. We show that then $B' = \emptyset$ or $B'' = \emptyset$.

We choose some $\alpha_0 \in \Gamma$. Let $e_{\alpha_0} \in B'$. Let $\alpha \in \Gamma \setminus \{\alpha_0\}$.

If $e_\alpha \in B''$ then $e_\alpha + e_{\alpha_0} \in B$ can be neither in B' nor in B'' , because in both cases $\text{span } B' \cap \text{span } B'' \neq \{0\}$. Therefore $e_\alpha \in B'$, hence $B \subset \text{lin } B'$ i.e. $B'' = \emptyset$.

Definition 2 (Dyer, [2], p.53-55). Let a B -space X have a gMb (B, γ) . Let $\gamma' = \gamma^* \setminus \{0\}$, where γ^* is the w^* -closure of γ in X . Then (B, γ') is a gMb of X , γ' -orthogonality is γ -orthogonality and γ' endowed with the partialized w^* -topology from X^* is a Boolean space which we denote by H . We call the Dyer embedding of X into $C_0(H)$ the mapping $T: X \rightarrow C_0(H)$ defined $Tx(f) = f(x)$ for any $f \in \gamma'$, $x \in X$. Dyer shows in [2] that T is continuous, linear, one-to-one and onto a dense subset of $C_0(H)$.

Remark 2. If (B, γ) is a Mb, then $C_0(H) = c_0(H)$ and so X has an equivalent strictly convex norm (see [1]). Because $l_\infty(\Gamma)$ for Γ infinite has no equivalent strictly convex norm ([1]), $l_\infty(\Gamma)$ is an example of a B -space which has a gMb and has no Mb. Our aim is to find the necessary and sufficient condition for w^* - w -continuity of the Dyer embedding of a B -space X^* with a gMb into $C_0(H)$. We need the following lemmas and definitions.

Definition 3. Suppose X has a gMb (B, γ) . Then we denote by $l_0(B)$ the following B -spaces:
 $l_0(B) = \{\psi \in l_\infty(B);$ (a) there exist $C > 0$ so that for any finite γ -orthogonal set $D \subset B, \sum_{d \in D} |\psi(d)| \leq C;$
 (b) if $\sum_{i=1}^m \lambda_i b_i$ and $\sum_{j=1}^m \mu_j d_j$ are two (B, γ) -combinations and $\sum_{i=1}^m \lambda_i b_i = \sum_{j=1}^m \mu_j d_j$, then $\sum_{i=1}^m \lambda_i \psi(b_i) =$

$= \sum_{j=0}^m \mu_j y(d_j)$ as the vector subspace of $\ell_\infty(B)$ and with the norm $\| \cdot \|_0$ defined $\| y \|_0 = \sup \{ \sum_{d \in D} |y(d)| \}$; D is a finite γ -orthogonal subset of B ; for any $y \in \ell_0(B)$.

Lemma 1. $\ell_0(B)$ is a B -space.

Proof: Easy.

Lemma 2. $\ell_0(B)$ is isometrically isomorphic with $[C_0(H)]^*$.

Proof. Let $A = \{ \sum_{i=0}^m \lambda_i T(b_i); \sum_{i=0}^m \lambda_i b_i \}$ is a (B, γ) -combination, where T is the Dyer embedding of X into $C_0(H)$, A is a normed linear space (with the norm $\| \cdot \|$) with the same dual as $C_0(H)$. For any $y \in \ell_0(B)$, $x \in A$ we define $\langle x, y \rangle = \sum_{i=0}^m \lambda_i y(b_i)$, where $x = \sum_{i=0}^m \lambda_i T(b_i), \sum_{i=0}^m \lambda_i b_i$ is a (B, γ) -combination. The definition of $\langle \cdot, \cdot \rangle$ is correct because if $\sum_{i=0}^m \lambda_i b_i$ and $\sum_{j=0}^m \mu_j d_j$ are two (B, γ) -combinations, then: $\sum_{i=0}^m \lambda_i T(b_i) = \sum_{j=0}^m \mu_j T(d_j)$ implies $\sum_{i=0}^m \lambda_i b_i = \sum_{j=0}^m \mu_j d_j$ and so $\sum_{i=0}^m \lambda_i y(b_i) = \sum_{j=0}^m \mu_j y(d_j)$ as follows from the condition (b). We show that $\langle A, \ell_0(B) \rangle$ is a dual pair and that in this duality $\ell_0(B)$ represents all elements of A^* , where A^* is the dual of A . By means of (b) and properties of T we show that $\langle \cdot, \cdot \rangle$ is bilinear. Obviously $\langle \cdot, \cdot \rangle$ is separated. Let $y \in \ell_0(B)$, $x \in A, x = \sum_{i=0}^m \lambda_i T(b_i), \sum_{i=0}^m \lambda_i b_i$ is the (B, γ) -combination. Then:

$$|\langle x, y \rangle| = \left| \sum_{i=0}^m \lambda_i y(b_i) \right| \leq \max_{i=1, \dots, m} |\lambda_i| \cdot \sum_{i=0}^m |y(b_i)| \leq \|x\| \cdot \|y\|_0$$

because $\|x\| = \max |\lambda_i|$ as it is easily seen from the fact that $T(b_i) = \chi_{E(b_i)}$, where $E(b_i) = \{f \in \mathcal{F}; f(b_i) = 1\}$, $i = 0, 1, \dots, m$, are pairwise disjoint and $\chi_{E(b_i)}$ denotes the characteristic function of $E(b_i)$. Hence any $y \in \ell_0(B)$ represents the unique $\mu_y \in A^*$. Moreover, it is easy to prove that $\|\mu_y\|_* = \|y\|_0$, where $\|\cdot\|_*$ is the dual norm in A^* .

Let $\mu \in A^*$. For any $b \in B$ we define $y(b) = \mu(Tb)$. Then y evidently satisfies (b) and if $\{b_0, \dots, b_m\} \subset B$ is γ -orthogonal, then

$$\begin{aligned} \sum_{i=0}^m |y(b_i)| &= \sum_{i=0}^m |\mu(Tb_i)| = \sum_{i=0}^m a_i \cdot \mu(Tb_i) = \\ &= \mu\left(\sum_{i=0}^m a_i Tb_i\right) \leq \|\mu\|_* \cdot \sum_{i=0}^m a_i, \text{ where } a_i = \text{sign } \mu(Tb_i), \end{aligned}$$

for any $i = 0, 1, \dots, m$. Hence $y \in \ell_0(B)$. Obviously $\mu = \mu_y$. Thus A^* is isometrically isomorphic with $\ell_0(B)$.

Theorem. Let a B -space X^* have a $g\mathbb{M}b(B, \gamma)$, let T be the Dyer embedding of X^* into $C_0(H)$. Then T is w^* - w -continuous iff each $F \in X^{**}$ satisfying the condition:

$$(1) \sup \left\{ \sum_{d \in D} |F(d)|; D \subset B \text{ is } \gamma\text{-orthogonal} \right\} < +\infty$$

is an element of $\mathfrak{K}(X)$.

Proof. For any $\mu \in A^*$ we denote by $\tilde{\mu}$ the continuous extension of μ on $C_0(H)$.

T is w^* - w -continuous iff for any $\mu \in A^*$ it holds

$\tilde{\mu} T \in \mathfrak{X}(X)$. From Lemma 2 it follows that T is w^* - w -continuous iff for any $\eta \in \ell_0(B)$ $\tilde{\mu}_\eta T \in \mathfrak{X}(X)$.

a) Suppose that for any $\eta \in \ell_0(B)$, $\tilde{\mu}_\eta T \in \mathfrak{X}(X)$. Let $F \in X^{**}$ satisfy (1). Then $F|_B$ -restriction of F to B is an element of $\ell_0(B)$. For any $b \in B$ $F(b) = (\mu_{F|_B}(Tb))$ and so, because $\text{sp } B = X^*$, $F = \tilde{\mu}_{F|_B} T \in \mathfrak{X}(X)$.

b) Suppose that for any $F \in X^{**}$ satisfying (1) it is $F \in \mathfrak{X}(X)$.

Let $\eta \in \ell_0(B)$. We can extend η linearly continuously on the element $F_\eta \in X^{**}$ satisfying (1). Hence

$$\tilde{\mu}_\eta T = \tilde{\mu}_{F|_B} T = F_\eta \in \mathfrak{X}(X).$$

Corollary 1. Let X^* have a Mb (B, γ) , T is the Dyer embedding of X^* into $c_0(H)$. Then T is w^* - w -continuous iff $\gamma \subset \mathfrak{X}(X)$.

Proof. a) Let T be w^* - w -continuous and $F \in \gamma$. Then $\sum_{b \in B} |F(b)| = 1$, so F satisfies (1) and hence $F \in \mathfrak{X}(X)$.

b) Let $\gamma \subset \mathfrak{X}(X)$. Let $F \in X^{**}$ satisfy (1). Since any finite subset of B is γ -orthogonal, it holds that $\sum_{b \in B} |F(b)| < +\infty$ and so $F = \sum_{b \in B} F(b) \cdot F_b$, where $F_b \in \gamma$ so that $F_b(b) = 1$, for any $b \in B$. Hence $F \in \mathfrak{X}(X)$.

Corollary 2. Let X^* have a Mb (B, γ) so that $\gamma \subset \mathfrak{X}(X)$. Then X is weakly compactly generated.

Proof. The Dyer embedding is then w^* - w -continuous. The closed unit ball in X^* is w^* -compact and hence affinely homeomorphic with some w -compact subset of $c_0(H)$.

From [3], Theorem 3.3 it follows that X is weakly compactly generated.

R e f e r e n c e s

- [1] M.M. DAY: Strict convexity and smoothness of normed spaces, Trans.Amer.Math.Soc.78(1955),516-528.
- [2] J.A. DYER: Generalized Markushevich bases, Israel Journ. Math.7(1969),51-59.
- [3] J. LINDENSTRAUSS: Weakly compact sets, their topological properties and Banach spaces they generate, Proc.Symp.Infinite Dim.Topology 1967, Ann. Math.Studies,Princeton,N.J.No 69(1972).
- [4] J. LINDENSTRAUSS: Decomposition of Banach spaces, Indiana Univ.Journ.20(1971),917-919.

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(Oblatum 24.4. 1973)