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MAXIMUM MODULUS FUNCTION OF DERIVATIVES OF ENTIRE FUNCTIONS DEFINED BY DIRICHLET SERIES

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Abstract: Let E be the set of mappings $f : C \to C$ (C is the complex field) such that the image under f of a point $s \in C$ is

$$f(s) = \sum_{n \in N} a_n e^{s\lambda_n} \quad \text{with} \quad \limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = D \in R_+ \cup \{0\}$$

(R_+ is the set of positive reals), and $\sigma_c^f = +\infty$. Then f is an entire function and is bounded on each vertical line $\mathrm{Re}(s) = \sigma_0$.

Denoting the maximum modulus function of the p-th derivative $f^{(p)}$ of f for any $p \in Z_+$ (Z_+ is the set of positive integers) by M_p, we have investigated into some of its properties. In particular, we have shown that, for the members of a certain subset of E, the functions M_p and M_{p+1} are separated from each other by the derivative of the former for sufficiently large values of σ.

Key words: Entire function, Dirichlet series, maximum modulus function, Ritt order, lower order, convex function, ordinary proximate linear order.

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1. Let E be the set of mappings $f : C \to C$ (C is the complex field) such that the image under f of an element $s \in C$ is

$$f(s) = \sum_{n \in N} a_n e^{s\lambda_n} \quad \text{with} \quad \limsup_{n \to +\infty} \frac{\log n}{\lambda_n} = D \in R_+ \cup \{0\}$$

($\mathbb{R}_+$ is the set of positive reals), and $\sigma_c^f = +\infty$ (σ_c^f is the abscissa of convergence of the Dirichlet series defining f); N is the set of natural numbers $0, 1, 2, \ldots$, $\langle a_m \mid m \in N \rangle$ is a sequence in C , $s = \sigma + it$, $\sigma, t \in \mathbb{R}$ ($\mathbb{R}$ is the field of reals), and $\langle \lambda_m \mid m \in N \rangle$ is a strictly increasing unbounded sequence of nonnegative reals. Since the Dirichlet series defining f converges for each $s \in C$, f is an entire function. Also, since $D \in \mathbb{R}_+ \cup \{0\}$, we have ([1], p. 168) $\sigma_a^f = +\infty$ (σ_a^f is the abscissa of absolute convergence of the Dirichlet series defining f), and that f is bounded on each vertical line $\mathrm{Re}(s) = \sigma_0$.

For any $p \in N$, the maximum modulus function M_p of the p -th derivative $f^{(p)}$ of an entire function $f \in E$, on any vertical line $\mathrm{Re}(s) = \sigma$, is defined as

$$(1.1) \qquad M_p(\sigma, f^{(p)}) = \sup_{t \in R} \{ |f^{(p)}(\sigma + it)| \}, \forall \sigma < \sigma_c^f .$$

We denote the function M_0 by M and study a few properties of the function M_p in this paper.

2. For every entire function $f \in E$, Doetsch has shown ([2], p.240) that $\log M_p$ is a downward convex function of σ . We may, therefore, write, for any σ , $\sigma_0 \in \mathbb{R}$ such that $\sigma > \sigma_0$,

(2.1) $$\log M_p(\sigma, f^{(p)}) = O(1) + \int_{\sigma_0}^{\sigma} V(x, f^{(p)})\,dx \quad ,$$

where V is a real valued indefinitely increasing function of σ, and establish

<u>Theorem 1</u>. <u>If the function</u> M_p <u>assumes the value unity at the origin, and</u> $\sigma_1, \sigma_2, \ldots, \sigma_n \in R$ <u>are such that</u> $0 \le \sigma_1 \le \sigma_2 \le \ldots \le \sigma_n$, <u>then</u>

(2.2) $$\frac{1}{n} \sum_{1\le k\le n} \sigma_k V(\sigma_k, f^{(p)}) \ge \sigma_1 V(0, f^{(p)}) +$$

$$+ \sum_{2\le k\le n} [k\sigma_k - (\sigma_1 + \ldots + \sigma_k) V(\frac{\sigma_1 + \ldots + \sigma_{k-1}}{k-1}, f^{(p)})] \ .$$

<u>Proof</u>. Using the convexity property of $\log M_p$, we get

$$\log M_p(\frac{m_1\sigma_1 + \ldots + m_n\sigma_n}{m_1 + \ldots + m_n}, f^{(p)}) \le \frac{m_1 \log M_p(\sigma_1, f^{(p)}) + \ldots + m_n \log M_p(\sigma_n, f^{(p)})}{m_1 + \ldots + m_n}$$

or

$$M_p(\frac{m_1\sigma_1 + \ldots + m_n\sigma_n}{m_1 + \ldots + m_n}, f^{(p)}) \le ((M_p(\sigma_1, f^{(p)}))^{m_1} \ldots (M_p(\sigma_n, f^{(p)}))^{m_n})^{1/(m_1 + \ldots + m_n)},$$

or, supposing $m_1 = \ldots = m_n = 1$,

(2.3) $$M_p(\frac{\sigma_1 + \ldots + \sigma_n}{n}, f^{(p)}) \le (M_p(\sigma_1, f^{(p)}) \ldots$$

$$\ldots M_p(\sigma_n, f^{(p)}))^{1/n} \ .$$

Since (2.1) is true for any $\sigma, \sigma_0 \in R$, we take $\sigma_0 = 0$

and $\sigma_0 = \sigma_n$ getting

$$(2.4) \qquad \log M_p(\sigma_n, f^{(p)}) = \int_0^{\sigma_n} V(x, f^{(p)})\,dx$$

$$\leq \sigma_n V(\sigma_n, f^{(p)}) ,$$

and so $M_p(\sigma_n, f^{(p)}) \leq \exp(\sigma_n V(\sigma_n, f^{(p)}))$. Therefore,

$$(2.5) \qquad (M_p(\sigma_1, f^{(p)}) \ldots M_p(\sigma_n, f^{(p)}))^{1/n} \leq$$

$$\leq \exp(n^{-1}(\sigma_1 V(\sigma_1, f^{(p)}) + \ldots + \sigma_n V(\sigma_n, f^{(p)}))) .$$

Again, from (2.4),

$$\log M_p\left(\frac{\sigma_1 + \ldots + \sigma_n}{n}, f^{(p)}\right) = \int_0^{(\sigma_1+\ldots+\sigma_n)n^{-1}} V(x, f^{(p)})\,dx$$

$$= \left(\int_0^{\sigma_1} + \int_{\sigma_1}^{(\sigma_1+\sigma_2)2^{-1}} + \ldots + \int_{(\sigma_1+\ldots+\sigma_{n-1})(n-1)^{-1}}^{(\sigma_1+\ldots+\sigma_n)n^{-1}}\right) V(x, f^{(p)})\,dx$$

$$(2.6) \qquad \geq \sigma_1 V(0, f^{(p)}) + \frac{2\sigma_2 - (\sigma_1 + \sigma_2)}{2 \cdot 1} V(\sigma_1, f^{(p)}) + \ldots +$$

$$+ \frac{n\sigma_n - (\sigma_1 + \ldots + \sigma_n)}{n(n-1)} V\left(\frac{\sigma_1 + \ldots + \sigma_{n-1}}{n-1}, f^{(p)}\right) .$$

But, from (2.4), we have

$$(2.7) \qquad \log M_p(\sigma_1, f^{(p)}) \geq \sigma_1 V(0, f^{(p)}) .$$

The theorem, therefore, follows from (2,3),(2.5),(2.6) and (2.7).

3. We next study two theorems concerning the derivative of

Theorem 2. *If $f \in E$ is an entire function of Ritt order $\rho \in R_+$ and lower order $\lambda \in R_+$ such that $\lambda \geq \sigma > 0$, then for any $\varepsilon \in R_+$, such that $\varepsilon = \varepsilon(\sigma, f)$ tends to zero as σ tends to plus infinity, and sufficiently large σ,*

$$(3.1)\quad M'_p(\sigma, f^{(p)}) > M'(\sigma, f)\left(\frac{\log M'(\sigma, f)}{(\frac{1}{\lambda} - \frac{1}{\rho} + \varepsilon)\log \lambda_{\nu(\sigma, f^{(p)})}}\right)^p,$$

and

$$(3.2)\quad M'_p(\sigma, f^{(p)}) > M'(\sigma, f)\left(\frac{\log M'(\sigma, f)}{(1 - \frac{\lambda}{\rho} + \varepsilon)}\right)^p ;$$

$\nu(\sigma, f^{(p)})$ is the rank of the maximum term $\mu(\sigma, f^{(p)})$ in the Dirichlet series defining $f^{(p)}$.

The proof of this theorem is based on the following lemmas.

Lemma 1. For every entire function $f \in E$ of Ritt order $\rho \in R_+$ and lower order $\lambda \in R_+$,

$$(3.3)\quad \limsup_{\sigma \to +\infty} \frac{\log M(\sigma, f)}{\lambda_{\nu(\sigma, f)} \log \lambda_{\nu(\sigma, f)}} \leq \frac{1}{\lambda} - \frac{1}{\rho},$$

and

$$(3.4)\qquad \limsup_{\sigma\to+\infty} \frac{\log M(\sigma,f)}{\sigma\lambda_{\nu(\sigma,f)}} \leq 1-\frac{\lambda}{\rho} .$$

The proof follows from the facts ([3], Theorem 5) that, as $\sigma \to +\infty$

$$(3.5)\qquad \log \mu(\sigma,f) \sim \log M(\sigma,f) ,$$

and ([4], p.57) that

$$(3.6)\qquad \limsup_{\sigma\to+\infty} \frac{\log \mu(\sigma,f)}{\lambda_{\nu(\sigma,f)} \log \lambda_{\nu(\sigma,f)}} \leq \frac{1}{\lambda}-\frac{1}{\rho} ,$$

and

$$(3.7)\qquad \limsup_{\sigma\to+\infty} \frac{\log \mu(\sigma,f)}{\sigma\lambda_{\nu(\sigma,f)}} \leq 1-\frac{\lambda}{\rho} .$$

Lemma 2 ([5], p.254). For every entire function $f \in E$,

$$(3.8)\qquad M_p(\sigma, f^{(p)}) \geq M'_{p-1}(\sigma, f^{(p-1)}) .$$

Proof of Theorem 2. We know ([6], p.67) that $\log \mu$ and $\log M$ are convex functions and (3.5) holds. It, therefore, follows that, as $\sigma \to +\infty$,

$$\frac{\mu'(\sigma,f)}{\mu(\sigma,f)} \sim \frac{M'(\sigma,f)}{M(\sigma,f)} .$$

But ([7], p.241), for almost all values of σ ,

$$\frac{\mu'(\sigma, f)}{\mu(\sigma, f)} = \lambda_{\nu(\sigma, f)} . \tag{3.9}$$

Hence, for almost all sufficiently large σ ,

$$\frac{M'(\sigma, f)}{M(\sigma, f)} \sim \lambda_{\nu(\sigma, f)} . \tag{3.10}$$

From (3.3) and (3.10) we get

$$\limsup_{\sigma \to +\infty} \frac{\log M(\sigma, f)}{\frac{M'(\sigma, f)}{M(\sigma, f)} \log \lambda_{\nu(\sigma, f)}} \leq \frac{1}{\lambda} - \frac{1}{\rho} .$$

Hence, for any $\varepsilon \in \mathbb{R}_+$ and sufficiently large σ ,

$$\frac{M'(\sigma, f)}{M(\sigma, f)} > \frac{\log M(\sigma, f)}{(\frac{1}{\lambda} - \frac{1}{\rho} + \varepsilon) \log \lambda_{\nu(\sigma, f)}} . \tag{3.11}$$

Writing (3.11) for $f^{(k)}$ we get

$$\frac{M_k'(\sigma, f^{(k)})}{M_k(\sigma, f^{(k)})} > \frac{\log M_k(\sigma, f^{(k)})}{(\frac{1}{\lambda} - \frac{1}{\rho} + \varepsilon) \log \lambda_{\nu(\sigma, f^{(k)})}} ,$$

whence, in view of Lemma 2,

$$\frac{M_k'(\sigma, f^{(k)})}{M_{k-1}'(\sigma, f^{(k-1)})} > \frac{\log M_{k-1}'(\sigma, f^{(k-1)})}{(\frac{1}{\lambda} - \frac{1}{\rho} + \varepsilon) \log \lambda_{\nu(\sigma, f^{(k)})}} . \tag{3.12}$$

Giving k the values $1, 2, \dots, p$ in (3.12) and multiplying the resulting p inequalities, we get

$$(3.13)\ M'_p(\sigma, f^{(p)}) > \frac{M'(\sigma, f) \prod_{1 \le k \le p} \log M'_{k-1}(\sigma, f^{(k-1)})}{\left(\frac{1}{\lambda} - \frac{1}{\rho} + \varepsilon\right)^p \prod_{1 \le k \le p} \log \lambda_{\nu(\sigma, f^{(k)})}} .$$

But ([5], p.254), for sufficiently large σ,

$$M(\sigma, f) < M'(\sigma, f) < M'_1(\sigma, f^{(1)}) < M'_2(\sigma, f^{(2)}) < \dots ,$$

and ([8], p.708),

$$\lambda_{\nu(\sigma, f)} \le \lambda_{\nu(\sigma, f^{(1)})} \le \lambda_{\nu(\sigma, f^{(2)})} \le \dots .$$

Making use of these facts in (3.13), we get (3.1).

The proof of (3.2) is similar to that of (3.1) except that instead of (3.3) we have to use (3.4).

Corollary. Under the hypothesis of Theorem 2 and for sufficiently large σ,

$$M'_p(\sigma, f^{(p)}) > M'(\sigma, f)\left(\frac{\log M'(\sigma, f)}{\left(\frac{1}{\lambda} - \frac{1}{\rho} + \varepsilon\right)(\rho + \varepsilon)\sigma}\right)^p .$$

This follows from (3.1) and the following result in [9]:

$$\limsup_{\sigma \to +\infty} \frac{\log \lambda_{\nu(\sigma, f)}}{\sigma} = \rho .$$

The next theorem is interesting since it shows that for a certain class of entire Dirichlet series, the functions M_p and M_{p+1} are separated from each other by the derivative of the former for sufficiently large values of σ.

Theorem 3. Under the hypothesis of Theorem 2 and for sufficiently large

(3.14) $M_p(\sigma, f^{(p)}) < M'_p(\sigma, f^{(p)}) \leq M_{p+1}(\sigma, f^{(p+1)}),\ \forall\, p \in N$.

Proof. It is known ([10] ,Lemma 2) that for every entire function $f \in E$ of Ritt order $\rho \in R_+^* \cup \{0\}$ (R_+^* is the set of extended positive reals) and lower order $\lambda \in R_+^* \cup \{0\}$,

$$(3.15) \quad {}^{\rho}_{\lambda} = \lim_{\sigma \to +\infty} {}^{\sup}_{\inf} \frac{\log (M'(\sigma, f^{(p)}) / M(\sigma, f^{(p)})}{\sigma} ,$$

$\forall\, p \in N$.

Hence, for any $\varepsilon \in R_+$ and sufficiently large σ ,

$$M_p(\sigma, f^{(p)})\, e^{\sigma(\lambda - \varepsilon)} < M'_p(\sigma, f^{(p)}) .$$

Since ε is arbitrary and $\lambda \geq \delta > 0$, it follows that, for sufficiently large σ ,

$$(3.16) \qquad M_p(\sigma, f^{(p)}) < M'_p(\sigma, f^{(p)}) .$$

Combining (3.16) with (3.8), we get (3.14).

4. Finally, we establish a result regarding ordinary proximate linear order of entire functions in E of irregular growth. We first recall its definition.

Definition ([11], p.112): A nonnegative extended real valued function Φ of reals σ is called an ordinary proximate linear order of an entire function $f \in E$ of Ritt order $\rho \in R_+$ provided

(a) Φ is eventually a continuous function,

(b) Φ is differentiable almost everywhere except at the isolated points at which the left and right derivatives exist,

(c) $\lim_{\sigma \to +\infty} \sigma \Phi'(\sigma) = 0$,

(d) $\limsup_{\sigma \to +\infty} \Phi(\sigma) = \rho$, and

(e) $\limsup_{\sigma \to +\infty} \dfrac{\log M(\sigma, f)}{e^{\sigma \Phi(\sigma)}} = 1$.

Theorem 4. *For every entire function $f \in E$ of irregular growth of Ritt order $\rho \in R_+$ and lower order $\lambda \in R_+$ and ordinary proximate linear order Φ ,*

$$\liminf_{\sigma \to +\infty} \frac{\log M_p(\sigma, f^{(p)})}{e^{\sigma \Phi(\sigma)}} = 0 . \tag{4.1}$$

Proof. We know that

$$\lim_{\sigma \to +\infty} \begin{matrix} \sup \\ \inf \end{matrix} \frac{\log M_p(\sigma, f^{(p)})}{\sigma} = \begin{matrix} \rho \\ \lambda \end{matrix} . \tag{4.2}$$

From (4.2), we get, for any $\varepsilon \in R_+$ and sufficiently large values of σ ,

$$(4.3) \qquad \log M_p(\sigma, f^{(p)}) > e^{(\lambda-\varepsilon)\sigma} ,$$

and, for an infinite sequence of σ's,

$$(4.4) \qquad \log M_p(\sigma, f^{(p)}) < e^{(\lambda+\varepsilon)\sigma}$$

Dividing (4.3) and (4.4) by $e^{\sigma\phi(\sigma)}$ and proceeding to limit we get (4.1) in view of the condition (d) of the definition of ordinary proximate linear order.

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