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A REPRESENTATION OF MODELS OF PEANO ARITHMETIC

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Abstract: The following theorem is proved: algebraically closed field of char. 0 is saturated if and only if every countable model of Peano arithmetic can be embedded into it.

Key words: Peano arithmetic, algebraically closed field of char. 0, saturated model, embedding.

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Introduction. In this paper, we shall present some results on embeddability of countable models of Peano arithmetic $\mathcal{P}$ into models of algebraically closed fields of characteristics 0.

This set-theoretical result should be compared with (and was inspired by) a recent result of Vopěnka saying that (under reasonable assumptions on existence of semi-sets) each countable model of $\mathcal{P}$ can be embedded into the field of real numbers by a semi-set embedding.

§ 0. Notations. Let $\mathcal{U}$ and $\mathcal{L}$ be structures of the same language. $\mathcal{h}: \mathcal{U} \rightarrow \mathcal{L}$ denotes that $\mathcal{h}$ is an embedding of $\mathcal{U}$ into $\mathcal{L}$. By $\mathcal{U} \approx \mathcal{L}$ and $\mathcal{U} \equiv \mathcal{L}$ we mean that $\mathcal{U}$ is isomorphic to $\mathcal{L}$ and $\mathcal{U}$ is elementa-
rily equivalent to $L$ respectively. By $\mathcal{U} \subseteq L$ and $\mathcal{U} \preceq L$ we mean that $\mathcal{U}$ is a substructure of $L$ and $\mathcal{U}$ is an elementary substructure of $L$ respectively. $|\mathcal{U}|$ is the universe of $\mathcal{U}$, $T \mathcal{U}$ is the complete theory of $\mathcal{U}$. Given a structure $\mathcal{U}$ and theory $T$, $\mathcal{U} \models T$ means that $\mathcal{U}$ is a model of $T$.

$T_F$ ($T_{F_0}$) is the theory of fields ($T_F$ of char. 0), $ACF$ ($ACF_0$) is the theory of alg. closed fields ($ACP$ of char. 0).

The nonlogical symbols of $P$ are $0, 1, +, \cdot$. The predicate $<$ is defined by $x < y \equiv (\exists z \neq 0)(x + z = y)$.

Let $\mathcal{U}$ be the standard model of natural numbers. Let $I$ and $C$ be the structure of integers and complex numbers respectively.

If $\mathcal{U}$, $I$ are fields, $I \subseteq \mathcal{U}$ and $S \subseteq |\mathcal{U}|$, then $\mathcal{U}(S)$ is the least subfield of $\mathcal{U}$, containing $|I|$ and $S$. If $\mathcal{U}$ is a field then $\overline{\mathcal{U}}$ is the algebraic closure of $\mathcal{U}$.

§ 1. Main results.

1.1. **Theorem.** Let $\mathcal{U} \models ACF_0$. Then (1) iff (2).

(1) If $\mathcal{U} \models P$ and $\text{card} \mathcal{U} = \aleph_0$, then there is $h: \mathcal{U} \rightarrow I$.

(2) $I$ is saturated.

1.2. **Corollary.** If $\mathcal{U}$ is a countable model of $P$ and if $\mathcal{U}$ is an uncountable model of $ACF_0$ then $\mathcal{U}$ is embeddable into $I$.

1.3. **Corollary.** If $\mathcal{U}$ is a countable model of $P$ then $\mathcal{U}$ is embeddable into the field $C$ of complex numbers.
§ 2. **Basic lemmas**. The following theorems are well-known:

A) If \( \mathcal{K}, \mathcal{L} \) are fields, \( \mathcal{L} \subseteq \mathcal{K} \) and if \( S \) is a transcendence basis of \( \mathcal{K} \) over \( \mathcal{L} \) then \( \mathcal{K} \) is algebraic over \( \mathcal{L}(S) \) ([LA]).

B) Let \( \mathcal{K} \models TF \), \( \text{card } |\mathcal{U}| \geq \aleph_0 \). Then every algebraic extension of \( \mathcal{K} \) has the same cardinality as \( \mathcal{K} \) ([LA]).

C) Let \( \mathcal{K} \models TF \). Then \( \mathcal{K} \) is algebraic over \( \mathcal{K} \) ([LA]).

D) Let \( \mathcal{K} \models ACF \), Then \( \mathcal{K} \) is saturated iff \( \mathcal{K} \) has infinite transcendence degree over its prime subfield ([SA]).

E) Let \( \mathcal{U}, \mathcal{L} \) be saturated structures of the same cardinality and \( \mathcal{U} \models \mathcal{L} \). Then \( \mathcal{U} \models \mathcal{L} \) ([SA]).

F) Let \( \text{card } \mathcal{K} = \aleph_0 \) and let \( T \mathcal{K} \) be \( \omega \)-stable. Then there exists a saturated \( \mathcal{L} \supseteq \mathcal{K} \) of the same cardinality as \( \mathcal{K} \) ([SA]).

2.1. **Lemma**. If \( \mathcal{K} \models TF_\omega \) and if \( \text{card } \mathcal{K} > \aleph_0 \) then \( \mathcal{K} \) has infinite transcendence degree over its prime subfield.

**Proof.** Let \( \mathcal{K}_p \) be the prime subfield of \( \mathcal{K} \). Then \( \text{card } \mathcal{K}_p = \aleph_0 \). Let \( S \) be a transcendence basis of \( \mathcal{K} \) over \( \mathcal{K}_p \). By A) \( \mathcal{K} \) is algebraic over \( \mathcal{K}_p(S) \); by B) \( \text{card } \mathcal{K} = \text{card } \mathcal{K}_p(S) = \max (\aleph_0, \text{card } S) \).

It follows from this lemma and E) that any two un-
countable models of $\text{ACF}_0$ of the same cardinality are isomorphic.

2.2. **Lemma.** Let $\mathcal{L} \models \text{ACF}_0$ and let $\text{card } \mathcal{L} > \aleph_0$. Then there is a $\mathcal{L}' \subseteq \mathcal{L}$ such that $\mathcal{L}'$ is a saturated model of $\text{ACF}_0$ of cardinality $\aleph_0$.

**Proof.** Let $\mathcal{L}_p$ be the prime subfield of $\mathcal{L}$, and let $S$ be a transcendence basis of $\mathcal{L}$ over $\mathcal{L}_p$. Let $S' \subseteq S$, $\text{card } S' = \aleph_0$. By A) - D) $\mathcal{L}' = \mathcal{L}_p(S)$ has the required property.

§ 3. **Proof of Theorem 1.1.**

(i) (2) $\rightarrow$ (1). By 2.2, we can assume that $\mathcal{L}$ is countable. Let $\mathcal{U}_R$ be "rationals over $\mathcal{U}$". $T \mathcal{U}_R = \text{ACF}_0$ is $\omega$-stable. By F) there is an $\mathcal{U}' > \mathcal{U}_R$, such that $\mathcal{U}'$ is saturated and $\text{card } \mathcal{U}' = \text{card } \mathcal{U}_R = \aleph_0$. By E) $\mathcal{U} \cong \mathcal{L}'$. Let $\mathcal{U}$ be the isomorphism of $\mathcal{U}'$ onto $\mathcal{L}$. Then $\mathcal{U} \upharpoonright \mathcal{L}'$ is the required embedding.

(ii) (1) $\rightarrow$ (2). Let $\mathcal{U}$ be a countable model of $\mathcal{P}$ such that there is a sequence $\{ \mathcal{U}_i \}_{i \in \omega}$ of models of $\mathcal{P}$ such that

$$\mathcal{U} = \mathcal{U}_0 \supset \mathcal{U}_1 \supset \mathcal{U}_2 \supset \ldots \supset \mathcal{U}_i \supset \mathcal{U}_{i+1} \supset \ldots \subset \mathcal{U}$$

and let for each $i$, $a_i \in A_i - A_{i-1}$ and $\mathcal{U} = x < a_{i+1}$ for each $x \in A_i$.

For each $i$, let $I_i$ be "integers (positive, negative and zero) over $\mathcal{U}_i$". Evidently

$$(*) \quad 0 \neq x \in I_i \implies a_i \cdot x \in I_i - I_{i-1}$$

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Let $M, t \rightarrow T$. We prove that $\mathbb{F}(a_i)_{i \in \omega}$ is algebraically independent over $\mathbb{F}_p$.

Let $m$ be the first number such that there are $0 \neq b_{\sigma} \in |\mathbb{F}_p|$, $\sigma = 1, 2, \ldots, m$, satisfying the equality

$$\sum_{\sigma=1}^{m} b_{\sigma} \cdot \mathbb{F}(a_1) \cdots \mathbb{F}(a_m) = 0$$

(where $\mathbb{F}(a_\sigma) \in \mathbb{N}$). We can suppose that $b_{\sigma} \in |\mathbb{F}_p|$, where $\mathbb{F}_p$ are "integers of $\mathbb{F}_p$". Moreover, there is a $\sigma$ such that $\mathbb{F}(x) = 0$. Let $m$ be the extension of $\mathbb{F}_p$ onto $\mathbb{F}_0$. Then

$$\sum_{\sigma=1}^{m} m^{-1}(b_{\sigma}) \cdot a_1 \cdots a_m = 0$$

Put $m^{-1}(b_{\sigma}) = b_{\sigma}$ ($\neq 0$), then there is a $\sigma$ such that $\mathbb{F}(x) = 0$. Consequently

$$a_n \cdot \sum_{\sigma=1}^{m} b_{\sigma} \cdot a_1 \cdots a_m = - \sum_{\sigma=1}^{m} b_{\sigma} \cdot a_1 \cdots a_m$$

which contradicts to (\#).

References

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