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A REPRESENTATION OF MODELS OF PEANO ARITHMETIC

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Abstract: The following theorem is proved: algebraically closed field of char. 0 is saturated if and only if every countable model of Peano arithmetic can be embedded into it.

Key words: Peano arithmetic, algebraically closed field of char. 0 , saturated model, embedding.

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Introduction. In this paper, we shall present some results on embeddability of countable models of Peano arithmetic P into models of algebraically closed fields of characteristics 0 .

This set-theoretical result should be compared with (and was inspired by) a recent result of Vopěnka saying that (under reasonable assumptions on existence of semi-sets) each countable model of P can be embedded into the field of real numbers by a semi-set embedding.

§ 0. Notations. Let $\mathcal{A}$ and $\mathcal{B}$ be structures of the same language. $h : \mathcal{A} \rightarrow \mathcal{B}$ denotes that h is an embedding of $\mathcal{A}$ into $\mathcal{B}$. By $\mathcal{A} \approx \mathcal{B}$ and $\mathcal{A} \equiv \mathcal{B}$ we mean that $\mathcal{A}$ is isomorphic to $\mathcal{B}$ and $\mathcal{A}$ is elementarily equivalent to $\mathfrak{B}$ respectively. By $\mathfrak{A} \subseteq \mathfrak{B}$ and $\mathfrak{A} \prec \mathfrak{B}$ we mean that $\mathfrak{A}$ is a substructure of $\mathfrak{B}$ and $\mathfrak{A}$ is an elementary substructure of $\mathfrak{B}$ respectively. $|\mathfrak{A}|$ is the universe of $\mathfrak{A}$, $T\mathfrak{A}$ is the complete theory of $\mathfrak{A}$. Given a structure $\mathfrak{A}$ and theory T, $\mathfrak{A} \models T$ means that $\mathfrak{A}$ is a model of T. TF (TF_0) is the theory of fields (TF of char. 0), ACF (ACF_0) is the theory of alg. closed fields (ACF of char. 0).

The nonlogical symbols of P are $0, 1, +, \cdot$. The predicate $<$ is defined by $x < y \equiv (\exists z \neq 0)(x + z = y)$.

Let $\mathfrak{N}$ be the standard model of natural numbers. Let $\mathfrak{I}$ and $\mathfrak{C}$ be the structure of integers and complex numbers respectively.

If $\mathfrak{A}$, $\mathfrak{B}$ are fields, $\mathfrak{B} \subseteq \mathfrak{A}$ and $S \subseteq |\mathfrak{A}|$, then $\mathfrak{B}(S)$ is the least subfield of $\mathfrak{A}$, containing $|\mathfrak{B}|$ and S. If $\mathfrak{A}$ is a field then $\overline{\mathfrak{A}}$ is the algebraic closure of $\mathfrak{A}$.

§ 1. <u>Main results</u>.

1.1. <u>Theorem</u>. Let $\mathfrak{B} \models ACF_0$. Then (1) iff (2).

(1) If $\mathfrak{A} \models P$ and $card\, \mathfrak{A} = \aleph_0$, then there is $h: \mathfrak{A} \to \mathfrak{B}$.

(2) $\mathfrak{B}$ is saturated.

1.2. <u>Corollary</u>. If $\mathfrak{A}$ is a countable model of P and if $\mathfrak{B}$ is an uncountable model of ACF_0 then $\mathfrak{A}$ is embeddable into $\mathfrak{B}$.

1.3. <u>Corollary.</u> If $\mathfrak{A}$ is a countable model of P then $\mathfrak{A}$ is embeddable into the field $\mathfrak{C}$ of complex numbers.

§ 2. Basic lemmas. The following theorems are well-known:

A) If $\mathfrak{A}$, $\mathfrak{B}$ are fields, $\mathfrak{B} \subseteq \mathfrak{A}$ and if S is a transcendence basis of $\mathfrak{A}$ over $\mathfrak{B}$ then $\mathfrak{A}$ is algebraic over $\mathfrak{B}(S)$ ([LA]).

B) Let $\mathfrak{A} \models TF$, $card\, |\mathfrak{A}| \geq \aleph_0$. Then every algebraic extension of $\mathfrak{A}$ has the same cardinality as $\mathfrak{A}$ ([LA]).

C) Let $\mathfrak{A} \models TF$. Then $\overline{\mathfrak{A}}$ is algebraic over $\mathfrak{A}$ ([LA]).

D) Let $\mathfrak{A} \models ACF$, Then $\mathfrak{A}$ is saturated iff $\mathfrak{A}$ has infinite transcendence degree over its prime subfield ([SA]).

E) Let $\mathfrak{A}$, $\mathfrak{B}$ be saturated structures of the same cardinality and $\mathfrak{A} \equiv \mathfrak{B}$. Then $\mathfrak{A} \approx \mathfrak{B}$ ([SA]).

F) Let $card\, \mathfrak{A} = \aleph_0$ and let $T\mathfrak{A}$ be ω-stable. Then there exists a saturated $\mathfrak{B} \succ \mathfrak{A}$ of the same cardinality as $\mathfrak{A}$ ([SA]).

2.1. Lemma. If $\mathfrak{A} \models TF_0$ and if $card\, \mathfrak{A} > \aleph_0$ then $\mathfrak{A}$ has infinite transcendence degree over its prime subfield.

Proof. Let $\mathfrak{A}_P$ be the prime subfield of $\mathfrak{A}$. Then $card\, \mathfrak{A}_P = \aleph_0$. Let S be a transcendence basis of $\mathfrak{A}$ over $\mathfrak{A}_P$. By A) $\mathfrak{A}$ is algebraic over $\mathfrak{A}_P(S)$; by B) $card\, \mathfrak{A} = card\, \mathfrak{A}_P(S) = max(\aleph_0, card\, S)$.

It follows from this lemma and E) that any two uncountable models of ACF_0 of the same cardinality are isomorphic.

2.2. Lemma. Let $\mathcal{L} \models ACF_0$ and let $card\ \mathcal{L} > \aleph_0$. Then there is a $\mathcal{L}' \subseteq \mathcal{L}$ such that $\mathcal{L}'$ is a saturated model of ACF_0 of cardinality $\aleph_0$.

Proof. Let $\mathcal{L}_p$ be the prime subfield of $\mathcal{L}$, and let S be a transcendence basis of $\mathcal{L}$ over $\mathcal{L}_p$. Let $S' \subset S$, $card\ S' = \aleph_0$. By A) - D) $\mathcal{L}' = \overline{\mathcal{L}_p(S)}$ has the required property.

§ 3. Proof of Theorem 1.1.

(i) (2) $\longrightarrow$ (1). By 2.2, we can assume that $\mathcal{L}$ is countable. Let $\mathfrak{A}_R$ be "rationals over $\mathfrak{A}$". $T\overline{\mathfrak{A}}_R = ACF_0$ is ω-stable. By F) there is an $\mathfrak{A}' \succ \overline{\mathfrak{A}}_R$, such that $\mathfrak{A}'$ is saturated and $card\ \mathfrak{A}' = card\ \overline{\mathfrak{A}}_R = \aleph_0$. By E) $\mathfrak{A}' \approx \mathcal{L}$. Let h be the isomorphism of $\mathfrak{A}'$ onto $\mathcal{L}$. Then $h \restriction \mathfrak{A}'$ is the required embedding.

(ii) (1) $\longrightarrow$ (2). Let $\mathfrak{A}$ be a countable model of P such that there is a sequence $\{\mathfrak{A}_i\}_{i\in\omega}$ of models of P such that

$$\mathfrak{N} = \mathfrak{A}_0 \subsetneq \mathfrak{A}_1 \subsetneq \mathfrak{A}_2 \subsetneq \ldots \subsetneq \mathfrak{A}_i \subsetneq \mathfrak{A}_{i+1} \subsetneq \ldots \subset \mathfrak{A}$$

and let for each i, $a_i \in A_i - A_{i-1}$ and $\mathfrak{A} \models \underline{x} < \underline{a}_{i+1}$ for each $x \in A_i$.
For each i, let $\mathfrak{I}_i$ be "integers (positive, negative and zero) over $\mathfrak{A}_i$". Evidently

$$(*) \qquad 0 \neq x \in I_i \Longrightarrow a_i \cdot x \in I_i - I_{i-1} .$$

Let $h : \mathfrak{A} \longrightarrow \mathfrak{B}$. We prove that $h''\{a_i\}_{i\in\omega}$ is algebraically independent over $\mathfrak{B}_p$.

Let m be the first number such that there are $0 \neq b_j' \in |\mathfrak{B}_p|$, $j = 1, 2, \dots, m$, satisfying the equality

$$\sum_{j=1}^{m} b_j' \dots h(a_1)^{\nu_1(j)} \dots h(a_m)^{\nu_n(j)} = 0$$

(where $\nu_i(j) \in N$). We can suppose that $b_j' \in |\mathfrak{I}_p|$, where $\mathfrak{I}_p$ are "integers of $\mathfrak{B}_p$ ". Moreover, there is a j such that $\nu_m(j) = 0$. Let $\tilde{h}$ be the extension of $h \restriction \mathfrak{A}_0$ onto $\mathfrak{I}_0$. Then

$$\sum_{j=1}^{m} \tilde{h}^{-1}(b_j') \cdot a_1^{\nu_1(j)} \dots a_m^{\nu_m(j)} = 0 \ .$$

Put $\tilde{h}^{-1}(b_j') = b_j \ (\neq 0)$, then there is a j such that $\nu_m(j) = 0$. Consequently

$$a_m \cdot \sum_{j=1, \nu_m(j)\neq 0}^{m} b_j \cdot a_1^{\nu_1(j)} \dots a_m^{\nu_m(j)-1} = - \sum_{j=1, \nu_m(j)=0}^{m} b_j \cdot a_1^{\nu_1(j)} \dots a_m^{\nu_m(j)}$$

which contradicts to (*).

R e f e r e n c e s

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