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ON A HEAT POTENTIAL
(Preliminary communication)
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Abstract: In this note we deal with a heat potential and its boundary behaviour in connection with the Fourier problem for the heat equation in \( \mathbb{R}^2 \). For this purpose we define the so-called parabolic variation.

Key words: Potential, heat potential, double-layer potential, single-layer potential, parabolic variation, limits of potentials

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Let \( \Gamma \) be the well-known kernel in \( \mathbb{R}^2 \) defined by

\[
\Gamma(x,t) = \begin{cases} 
(\pi t)^{-\frac{n}{2}} \exp \left( -\frac{x^2}{4t} \right), & t > 0, \\
0, & t \leq 0,
\end{cases}
\]

and denote by \( \partial_x \Gamma \) its partial derivative with respect to the variable \( x \). Fix \( a \leq \mathcal{B} \) in \( \mathbb{R}^d \) and let \( C_0(<a,\mathcal{B}>) \) stand for the space of all continuous real-valued functions on \( <a,\mathcal{B}> \) vanishing at \( a \). Let \( \varphi \) be a fixed continuous function of bounded variation on \( <a,\mathcal{B}> \) and put

\[
\mathcal{X} = \{ \{ \varphi(t), t \}; a \leq t \leq \mathcal{B} \}.
\]
With each \( f \in C_0(\langle a, b \rangle) \) we shall associate the function \( T_f \) on \( \mathbb{R}^2 - K \) defined by

\[
T_f(x, t) = - \int_a^b f(\tau) \partial_t \Gamma(x - \varphi(\tau), t - \tau) d\tau - \int_a^b f(\tau) \Gamma(x - \varphi(\tau), t - \tau) d\varphi(\tau)
\]

for \( t > a \), \( T_f(x, t) = 0 \) for \( t \leq a \).

Investigation of \( T_f(x, t) \) (which is a combination of a double-layer and a single-layer heat potential) as \( \{x, t\} \) approaches \( K \) is of importance in connection with the boundary value problems for the heat equation (compare [7], § 4 in chap. VI; see also [1],[2],[6] for references concerning heat potentials). Our purpose in this note is to present a simple necessary and sufficient condition on \( K \) guaranteeing the existence of the finite limits

\[
\lim_{\{x, t\} \to \{x_0, t_0\}} T_f(x, t) = T_f(t_0), \quad \lim_{\{x, t\} \to \{x_0, t_0\}} T_f(x, t) = T_f(t_0)
\]

at \( \{x_0, t_0\} \in K \) for any \( f \in C_0(\langle a, b \rangle) \).

Given \( \{x_0, t_0\} \in \mathbb{R}^2 \) and \( \alpha > 0 \) consider the parabola

\[
P_\alpha(x_0, t_0) = \{ \{x, t\} \in \mathbb{R}^2 ; t_t - t = \left( \frac{x_0 - x}{2\alpha} \right)^2 \}
\]

and denote by \( n_\alpha(\alpha; x_0, t_0) \) the number of points in \( (K - \{x_0, t_0\}) \cap P_\alpha(x_0, t_0) \) (we put \( n_\alpha(\alpha; x_0, t_0) = + \infty \))
if the last set is infinite). Then \( m_\nu (\alpha; x_0, t_0) \) is a Lebesgue measurable extended-real-valued function of the variable \( \alpha \in (0, +\infty) \) and we may form the quantity

\[
V_\nu(x_0, t_0) = \int_0^\infty e^{-\alpha^2} m_\nu (\alpha, x_0, t_0) \, d\alpha
\]

to be termed the parabolic variation of \( X \) at \([x_0, t_0]\).

In connection with \( T \) the parabolic variation plays a role comparable with that of the so-called cyclic variation \( \nu^c \) as introduced in [3] in connection with the investigation of double-layer logarithmic potentials. The following theorem holds.

**Theorem.** If at least one of the limits (1) exists for every \( f \in C_0 (\langle a, b \rangle) \) then there is a \( \sigma > 0 \) such that

\[
\sup_{|t-t_0|<\sigma} V_\nu (\varphi(t), t) < \infty .
\]

Conversely, if (2) holds then the finite limits (1) exist for every bounded Baire function \( f \) on \( \langle a, b \rangle \) that is continuous at \( t_0 \) (and vanishes at \( a \) in case \( t_0 = a \)).

**Proof** of this theorem is based on the Banach theorem on variation of a continuous function and on ideas employed in [4] in connection with double-layer logarithmic potentials. The key part of the proof rests on the following lemma whose role is similar to that of Theorem 1.11 in [4].

**Lemma.** If (2) holds then there is a neighborhood \( U \) of \([\varphi(t_0), t_0]\) in \( \mathbb{R}^2 \) such that
If

\[ \sup_{[x, t] \in U} V_K(x, t) = +\infty. \]

then \( V_K(\cdot, \cdot) \) is bounded on the whole of \( \mathbb{R}^d \).

**Corollary.** \( T_f \) is uniformly continuous on each of the domains \( D^+_K, D^-_K \) for any \( f \in C_0(\langle a, b \rangle) \) if and only if (3) holds.

A modification of \( V_K \) permits to evaluate, in geometric terms, the Fredholm radius of the operators \( T_\omega + +(-1)^n I \) (where \( I \) is the identity operator on \( C_0(\langle a, b \rangle) \) and \( T_\omega \) are defined by (1)) and establish a general theorem on representability of the solution of the Fourier problem by means of \( T_f \). The applied methods have been worked up in [5]. The following assertion holds.

**Theorem.** Let the Fredholm radius of \( T_\omega - I \) (which we can express in geometric terms) be greater than 1. Put \( B = X \cup \{x, a, 1; x \geq \varphi(a)\} \) and let \( F \) be a continuous bounded function on \( B \) with \( F(\varphi(a), a) = 0 \). Then there is a unique function \( f \in C_0(\langle a, b \rangle) \) such that the function

\[ T_f(x, t) + \frac{1}{2\sqrt{\pi}} \int_{\varphi(a)}^\infty \frac{F(x, a)}{\sqrt{t-a}} e^{-\frac{x^2}{4(t-a)}} dx \]

is a solution of the Fourier problem on \( D^+_K \) for the boundary condition \( F \).
Analogical results may be obtained for the domain $D^*_K$ and for domains of the form $\{[x,t]; t \in (a, b), \varphi_1(t) < x < \varphi_2(t)\}$ where $\varphi_1$, $\varphi_2$ are some continuous functions of bounded variation on $(a, b)$ such that $\varphi_1(t) < \varphi_2(t)$ for each $t \in (a, b)$.

Complete proofs of these results together with further details and bibliography will be included in a paper which will be published in the Czechoslovak Mathematical Journal.

The following two assertions will be proved in a paper which will be published in Časopis pro pěstování matematiky.

**Theorem.** Let $t \in (a, b)$ and suppose that

$$\limsup_{x \to t-} \frac{|\varphi(t) - \varphi(x)|}{\sqrt{t-x}} < \infty.$$  

Then there is a finite limit

$$\lim_{x \to \varphi(t)+} T_f(x,t)$$

(or a finite limit

$$\lim_{x \to \varphi(t)-} T_f(x,t)$$

for any function $f \in C((a, b))$ if and only if

$$V_K(\varphi(t), t) < \infty.$$  

**Proposition.** There is a continuous function $\varphi$ of bounded variation on $(a, b)$ such that

$$V_K(\varphi(t), t) = \infty$$

for almost every $t \in (a, b)$ (where $K = \{[\varphi(t), t]; \quad t \in (a, b)\}$).
References


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