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PRINCIPAL SOLUTION OF THE DIRICHLET PROBLEM IN POTENTIAL THEORY

(Preliminary communication)

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Abstract: The new type of extension of the classical solution of the Dirichlet problem in the axiomatic potential theory is introduced. This one is, in general, different from the Perron solution. The details will appear in [4] .

Key words: Dirichlet problem, the generalized solution, principal solution

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Let us consider in the Euclidean space $\mathbb{R}^n$ the harmonic functions as the continuous solutions of the Laplace differential equation $\Delta f = 0$. For a bounded open set $\mathcal{U} \subset \mathbb{R}^n$ and for a continuous function f on the boundary $\mathcal{U}^*$ of $\mathcal{U}$, by $H_f^{\mathcal{U}}$ we denote a generalized solution of the Dirichlet problem for f obtained by the Perron method. We may construct the generalized solution also by other methods. For instance, by Wiener method we extend the given function f on the whole closure $\overline{\mathcal{U}}$ and we exhaust $\mathcal{U}$ by the regular sets for which we can solve the Dirichlet problem and we take the limit of such solutions in any point of $\mathcal{U}$. We obtain again the Perron solution $H_f^{\mathcal{U}}$ which folllows, for instance, from the following theorem of Keldych.

<u>Theorem</u> (M.V. Keldych 1941,[3]). Let $\mathcal{U}$ be an open bounded set in $\mathbb{R}^n$ and let Φ be a linear and monotone map associating with any continuous function f on $\mathcal{U}^*$ the harmonic function $\Phi(f)$ on $\mathcal{U}$ having the property that $\Phi(F) = F$ if F is continuous on $\overline{\mathcal{U}}$ and harmonic in $\mathcal{U}$. Then $\Phi(f)$ is equal to the Perron solution $H_f^{\mathcal{U}}$ for any continuous f on $\mathcal{U}^*$.

We mention still another method for the construction of the generalized solution $H_f^{\mathcal{U}}$. If p is a potential and $\mathcal{U} \subset \mathbb{R}^n$, we denote by $\hat{R}_p^{C\mathcal{U}}$ the balayage of p on the complement $C\mathcal{U}$ of $\mathcal{U}$. We may now formulate the following

<u>Proposition</u>. For any $x \in \overline{\mathcal{U}}$ there exists a unique Radon measure $\mu_x^{\mathcal{U}}$ on $\mathbb{R}^n$, whose support is contained in $\mathcal{U}^*$, such that $\mu_x^{\mathcal{U}}(p) = \hat{R}_p^{C\mathcal{U}}(x)$ for every potential p.

Let us observe that $\mu_x^{\mathcal{U}}(f) = H_f^{\mathcal{U}}(x)$ for any continuous f on $\mathcal{U}^*$ and for any $x \in \mathcal{U}$. According to this proposition we may extend the definition of $H_f^{\mathcal{U}}$ by means of the balayage to the points of the boundary $\mathcal{U}^*$ of $\mathcal{U}$. Therefore, for any $x \in \overline{\mathcal{U}}$ we understand $H_f^{\mathcal{U}}$ as a Radon measure on $\mathcal{U}^*$. The following theorem is important.

<u>Theorem</u>. If we denote for a continuous function f on $\mathcal{U}^*$ by F the restriction of $H_f^{\mathcal{U}}$ to $\mathcal{U}^*$, then F is a Borel function and $H_F^{\mathcal{U}} = H_f^{\mathcal{U}}$ on $\overline{\mathcal{U}}$.

The equality $\hat{R}^{C\mathcal{U}}_{\hat{R}_p^{C\mathcal{U}}} = \hat{R}_p^{C\mathcal{U}}$ for any potential p is essential for the proof of this theorem.

Let us consider now in the Euclidean space $\mathbb{R}^{n+1}$ the "harmonic" functions - some authors use the term "parabolic"

- as the continuous solutions of the heat equation $\Delta f = \frac{\partial f}{\partial t}$. In the same manner as for the Laplace equation we define the generalized solution $H_f^{\mathcal{U}}$ of the Dirichlet problem by the Perron method or equivalently by means of the balayaged functions. The direct application of the Wiener method is not useful here since an exhausting by regular sets need not always exist. However, we may construct in this case more "generalized solutions". The Keldych theorem is no more valid for the heat equation.
Likewise the equality $\hat{R}^{C\mathcal{U}}_{\hat{R}^{C\mathcal{U}}_p} = \hat{R}^{C\mathcal{U}}_p$ fails for the heat equation. This is caused by the fact that the role of polar sets of the Laplace equation is interchanged by semi-polar sets, which need not be so "small". For the same reason also the equality $H_f^{\mathcal{U}} = H_F^{\mathcal{U}}$ fails, where F denotes the restriction of $H_f^{\mathcal{U}}$ to $\mathcal{U}^*$.

The theory of harmonic functions derived from the Laplace equation or from the heat equation is a model for the general axiomatic theory of abstract harmonic spaces. In what follows we shall work in this theory. So $(X, \mathcal{H})$ is a strong harmonic space in the sense of the Bauer's axiomatic [1].

Let $\mathcal{U}$ be an open relatively compact subset of X. We denote again by $\hat{R}^{C\mathcal{U}}_p$ the balayage of p on $C\mathcal{U}$. For every $x \in \overline{\mathcal{U}}$ there exists a Radon measure $\mu_x^{\mathcal{U}}$, whose support lies in $\mathcal{U}^*$, such that $\hat{R}^{C\mathcal{U}}_p(x) = \mu_x^{\mathcal{U}}(p)$ for any continuous potential p. We put further $H_f^{\mathcal{U}}(x) = \mu_x^{\mathcal{U}}(f)$ for any Borel function f on $\mathcal{U}^*$ and for any $x \in \overline{\mathcal{U}}$. We may also obtain this generalized solution $H_f^{\mathcal{U}}$ by the Perron method and $\mu_x^{\mathcal{U}}$ for $x \in \mathcal{U}$ is then called the <u>harmonic measure</u> at x. For the sake of the general nonequality between $\hat{R}_p^{C\mathcal{U}}$ and $\hat{R}^{C\mathcal{U}}_{\hat{R}_p^{C\mathcal{U}}}$ or - what is the same - between $H_f^{\mathcal{U}}$ and $H_F^{\mathcal{U}}$, where F denotes the restriction of $H_f^{\mathcal{U}}$ to $\mathcal{U}^*$, we seek a new "balayage" and a new "generalized solution" without this lack. The construction is based on the following lemma.

<u>Lemma</u>. Let p be a potential on X. If $\mathcal{M}(p)$ is the set of all potentials q such that $q \leq p$ and $\hat{R}_q^{C\mathcal{U}} = q$, then the pointwise supremum of $\mathcal{M}(p)$ again belongs to it.

We then put $T_p^{C\mathcal{U}} = \sup\{q ; q \leq p, \hat{R}_q^{C\mathcal{U}} = q\}$ for any potential p. We know that $\hat{R}^{C\mathcal{U}}_{T_p^{C\mathcal{U}}} = T_p^{C\mathcal{U}}$. The potential $T_p^{C\mathcal{U}}$ is called the <u>principal balayage</u> of p on $C\mathcal{U}$. Again, the next proposition is crucial.

<u>Proposition</u>. For every $x \in \overline{\mathcal{U}}$ there exists a uniquely determined Radon measure $\nu_x^{\mathcal{U}}$ on X, whose support lies in $\mathcal{U}^*$, such that $\nu_x^{\mathcal{U}}(p) = T_p^{C\mathcal{U}}(x)$ for any continuous potential p on X.

Hence, we may put $L_f^{\mathcal{U}}(x) = \nu_x^{\mathcal{U}}(f)$ for any continuous function f on $\mathcal{U}^*$ and for any $x \in \overline{\mathcal{U}}$. The function $L_f^{\mathcal{U}}$ defined on $\overline{\mathcal{U}}$ is termed the <u>principal solution</u> of the Dirichlet problem for f. The essential properties of the principal solution are stated in the following assertions.

<u>Proposition</u>. The function $L_f^{\mathcal{U}}$ is harmonic on $\mathcal{U}$, it is of first Baire class on $\overline{\mathcal{U}}$ and continuous on $\overline{\mathcal{U}}$ in the fine topology.

Theorem. If F denotes the restriction of $L_f^{\mathcal{U}}$ to $\mathcal{U}^*$, then $H_F^{\mathcal{U}} = L_f^{\mathcal{U}}$ on $\overline{\mathcal{U}}$.

Theorem. If s is a continuous function on $\overline{\mathcal{U}}$ which is superharmonic in $\mathcal{U}$, then $L_s^{\mathcal{U}} \leq H_s^{\mathcal{U}} \leq s$ on $\overline{\mathcal{U}}$. In particular, $L_h^{\mathcal{U}} = h$ on $\overline{\mathcal{U}}$ for any function h continuous on $\overline{\mathcal{U}}$ and harmonic in $\mathcal{U}$.

We see that the map $f \longmapsto L_f^{\mathcal{U}}$ is linear and monotone, that the function $L_f^{\mathcal{U}}$ is harmonic in $\mathcal{U}$ and that $L_h^{\mathcal{U}} = h$, if h is continuous on $\overline{\mathcal{U}}$ and harmonic in $\mathcal{U}$. In the case of the Laplace equation it follows from the Keldych theorem that $L_f^{\mathcal{U}} = H_f^{\mathcal{U}}$. We are interested to know whether this equality holds also in our general setting. First, we mention the generalization of the Keldych theorem due to M. Brelot (see [2]) if the additional domination axiom D is assumed. Let us observe that the classical solutions of the Laplace equation in $\mathbb{R}^n$ satisfy this axiom, while the solutions of the heat equation do not. If axiom D is fulfilled, then the set of all irregular points of $\mathcal{U}$ is polar and it is known that any polar set is of harmonic $\mu_x^{\mathcal{U}}$ -measure zero for every $x \in \mathcal{U}$. Now, our last theorem can be formulated as follows.

Theorem. The following assertions are equivalent:

(i) $H_f^{\mathcal{U}} = L_f^{\mathcal{U}}$ for any continuous function f ,

(ii) $\hat{R}_p^{c\mathcal{U}} = T_p^{c\mathcal{U}}$ for any continuous potential p ,

(iii) the set of all irregular points of $\mathcal{U}$ is of $\mu_x^{\mathcal{U}}$ -measure zero for every $x \in \mathcal{U}$.

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