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SOME PRIMITIVE CLASSES OF LATTICES CLOSED UNDER THE FORMATION OF PROJECTIVE IMAGES

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Abstract: In this paper it is shown that there exist infinitely many primitive classes $\mathcal{K}$ of lattices such that $L \in \mathcal{K}$, $Sub(L) \cong Sub(L')$ imply $L' \in \mathcal{K}$, where $Sub(L)$ denotes the lattice of all sublattices of L.

Key words: Lattice, primitive class, projective image.

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A lattice L' is said to be a projective image of a lattice L if there exists an isomorphism of $Sub(L)$ onto $Sub(L')$, where $Sub(L)$ denotes the set of all subsets of L closed under both meet and join; $Sub(L)$ is a lattice with respect to the set inclusion. G. Grätzer suggests (see [1], Problem 8) to find primitive classes of lattices which are closed under the formation of projective images. It is known (see [2],[4]) that the primitive class of all distributive, and the primitive class of all modular lattices, as well, has the property mentioned above. The purpose of this paper is to show that there exist infinitely many primitive classes of lattices closed under the formation of projective images.

Let L and L' be lattices and let ψ be an isomor-

phism of $\mathcal{Sub}(L)$ onto $\mathcal{Sub}(L')$. This isomorphism induces a bijection $\overline{\psi}$ of L onto L' defined by $\overline{\psi}(x) = y$ iff $\psi(\{x\}) = \{y\}$.
Since the elements x, y of L are comparable in L (i.e. $\{x, y\} \in \mathcal{Sub}(L)$; it has the length two in $\mathcal{Sub}(L)$) if and only if the elements $\overline{\psi}(x)$, $\overline{\psi}(y)$ are comparable in L', we have

Lemma 1. Let M be a lattice which is as lattice determined by the comparability relation uniquely up to isomorphism. Then $\mathcal{Sub}(M) \cong \mathcal{Sub}(M')$ implies $M \cong M'$.

Lemma 2. Let M be a lattice which is as lattice determined by the comparability relation uniquely up to isomorphism. Let L and L' be lattices such that $\mathcal{Sub}(L)$ is isomorphic to $\mathcal{Sub}(L')$ and let $M \in \mathcal{Sub}(L)$. Then $M \in \mathcal{Sub}(L')$.

Proof. Let ψ be an isomorphism of $\mathcal{Sub}(L)$ onto $\mathcal{Sub}(L')$. $\mathcal{Sub}(M)$ is a sublattice of $\mathcal{Sub}(L)$ and $\psi(\mathcal{Sub}(M)) = \mathcal{Sub}(\psi(M))$ is isomorphic to $\mathcal{Sub}(M)$. By Lemma 1 we get that $\psi(M)$ is isomorphic to M, i.e. $M \in \mathcal{Sub}(L')$.

Given a lattice L, we denote by L^* a lattice which is obtained from L by adding exactly three elements o, i, a; o is the smallest element of L^*, i is the greatest element of L^* and a is comparable with no element of L. The following Lemma 3 is evident.

Lemma 3. If a lattice L is as lattice determined by the comparability relation uniquely up to isomorphism then L^* has the same property.

Define two sequences of lattices by the following rules:

(i) L_1 is the five-element non-modular lattice;

(ii) M_1 is the five-element non-distributive modular lattice;

(iii) $L_{n+1} = L_n^*$ and $M_{n+1} = M_n^*$ for all $n \geq 1$.

It is easy to show that the lattices L_1 and M_1 are as lattices determined by the comparability relation uniquely up to isomorphism. By Lemma 3 we can get that the lattices L_n, M_n $(n \geq 1)$ have also this property. Given a lattice L, we denote by $K(L)$ the class of all lattices that contain no sublattice isomorphic to L. In the paper [3] it is proved that the classes $K(L_n)$ and $K(L_n) \cap \cap K(M_n)$ are primitive for all $n \geq 1$.

Combining this fact with Lemma 2 we get

<u>Theorem</u>. The primitive classes $K(L_n)$ and $K(L_n) \cap \cap K(M_n)$ (for all $n \geq 1$) are closed under the formation of projective images.

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