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ON A QUESTION OF PULTR REGARDING CATEGORIES OF STRUCTURES

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Abstract: It is known that every constructive structure can be realized as a structure based on a power (under composition) of the contravariant power-set functor. It is proved here that one can use the covariant one instead.

Key-words and phrases: Categories of structures, realize, majorize, covariant power set functor.

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Ales Pultr has given a definition which allows one to describe models of higher order theories in terms of first-order structures defined in the range of a functor from $\text{Set} \to \text{Set}$. This suggests the question: which functors generate structures comparable with those of ordinary nth order logic (for some $\alpha$)? Pultr has given a partial answer by finding a class of categories of models that can be realized in $\mathcal{S}(\langle \mathcal{P}^- \rangle^\omega \circ \mathcal{V}_A)$, the category of all models $(X, U)$ whose structure $U$ consists of a distinguished subset of $\langle \mathcal{P}^- \rangle^\omega \circ \mathcal{V}_A(X)$, where $\mathcal{P}^-$ is the usual contravariant power set functor and $\mathcal{V}_A$ is a sum of the identity functor and a constant functor. The present paper gives a similar partial answer by showing that these same categories can be realized in $\mathcal{S}(\langle \mathcal{P}^+ \rangle^\omega \circ \mathcal{V}_A)$, where $\mathcal{P}^+$...
is the usual covariant power set functor. As with Pultr’s work, if one is willing to allow infinite powers of $P^+$, then the class of functors involved can be enlarged by taking limits and colimits over small categories.

When not specified, the terminology is as in [1].

Set denotes the category of sets and functions. For any function $f: X \rightarrow Y$, let $f^\vee$ equal $(P^-)(f): P(Y) \rightarrow P(X)$, and let $f^\vee$ ambiguously represent $(P^+)^k(f): P^k(X) \rightarrow P^k(Y)$.

1 Lemma: $S((P^-)^2)$ is realizable in $S((P^+)^4)$; $(P^-)^2$ is majorized by $(P^+)^5$.

Proof. For any $U \subseteq P(X)$ and $A \subseteq X$, define $A$ to be $U$-substantial iff $\forall U \subseteq X, U \in U$ iff $U \cap A \in U$.

Step I: For any function $f: X \rightarrow Y$ and $U \subseteq P(X)$, if $A$ is $U$-substantial, then $f[A]$ is $f^\vee(U)$-substantial.

Since $f^\vee(U) = \forall V \subseteq Y; f^\vee(V) \subseteq U$, we have that $\forall V \subseteq Y, V \cap f[A] \subseteq f^\vee(U)$ iff $f^\vee(V \cap f[A]) = U$; but $f^\vee(V \cap f[A]) = f^\vee(V \cap f[A])$, and $f^\vee(V \cap f[A]) \in U$ iff $f^\vee(V \cap A) \in U$, iff $f^\vee(V) \in U$ iff $V \in f^\vee(U)$. Hence $f[A]$ is $f^\vee(U)$-substantial.

Define a functor $R: \text{Set} \rightarrow \text{Set}$ as follows: for any set $X$, $R(X)$ is the set of all pairs $\{X, Q\}$ such that

i) $X \in \{U: U \subseteq X\}$,

ii) $\emptyset \in U \cap Q$, $Q \in \{Q_1, Q_2\}: Q_1, Q_2 \subseteq X$ and $Q = \emptyset \in \{Q_1, Q_2\}$: $Q_1 = Q_2$ and $Q_1 = \emptyset \cup Q_2$, and $Q_1, Q_2 \in U \cap Q$,

iii) $\cup U \subseteq \cup U \cap Q$;
for any map $f: X \to Y$ let $\mathcal{R}(f) = (P^+)^4(f)$. By nonstandard convention, we shall consider phrases such as.

"$\{X, Q\} \in \mathcal{R}(X)$" to abbreviate "$\{X, Q\} \in \mathcal{R}(X)$, $X$ satisfies (i), and $Q$ satisfies (ii)".

Step II: If $f: X \to Y$, $\{X, Q\} \in \mathcal{R}(X)$, $\{Y, R\} \in \mathcal{R}(Y)$, and $f^\sim(\{X, Q\}) = \{Y, R\}$, then $f^\sim(X) = Y$ and $f^\sim(Q) = R$.

Suppose not; then $f^\sim(Q) = Y$ and $f^\sim(X) = R$. Now if $\cup U(Q)$ were non-empty, $f^\sim(Q)$ would contain a nontrivial pair of the form $\{\emptyset, f[Q]\}$. But $Y$ contains only singletons. Hence $Q = \{\emptyset\}$ since $\emptyset \in U(Q)$. Consequently $f^\sim(Q) = \{\emptyset\}$. Similarly, $\cup U(f^\sim(X)) = \cup U(R)$ must be empty, so that $R = \{\emptyset\} = X$. Hence $f^\sim(X) = Y$ and $f^\sim(Q) = R$.

For any $\{X, Q\} \in \mathcal{R}(X)$, define $Q$ to be significant iff $\forall Q_1, Q_2 \in Q$, $Q_1 \cap Q_2 = \emptyset$.

Step III: It is easy to see that given $f: X \to Y$ and $\{X, Q\} \in \mathcal{R}(X)$, $f^\sim(Q)$ is significant iff $Q$ is significant and $\forall Q_1, Q_2 \in Q$, $Q_1 \neq Q_2$ implies $f[Q_1] \cap f[Q_2] = \emptyset$.

A realization of $S((P^-)^2)$ in $S(R)$ can now be given as follows: for each $X$ and $U \in P^2(X)$, let $U^*$ be the set of all $\{X, Q\} \in \mathcal{R}(X)$ such that if $Q$ is significant, then for some $U \in U$, $UUQ$ is $U$-substantial and $U\mathcal{R} = \{U \in U \mid \exists Q \subseteq UQ, U = UQ\}$. Let $f: X \to Y$, $U \in P^2(X)$, and $V \in P^2(Y)$ be arbitrary.

Step IV: If $K(f)[U^*] \subseteq V^*$, then $f^\sim[V] \subseteq V$. Pick $U \in U$. Let $Q$ be the set of all pairs $\{f^\sim(A), f^\sim(B)\}$.
such that $A, B \subseteq Y$, $A \cap B = \emptyset$, and $\text{card } A, \text{card } B \leq 1$.

Let $\mathcal{X} = \{U \in \mathcal{U} : U \in \mathcal{U} \text{ and } \exists A \subseteq U \in \mathcal{Q}, U = UA\}$. Then $\{X, Q\} \in \mathcal{U}^*$, and thus $f^\sim(X, Q) \in \mathcal{U}^*$, $f^\sim(Q)$ is clearly significant, and thus we may choose $V \in \mathcal{V}$ so that $UUf^\sim(Q) \in \mathcal{V}$ -substantial and $Uf^\sim(X) = \{V \in \mathcal{V} : \exists B \subseteq Uf^\sim(Q), V = UB\}$. We need to show $V = f^\sim(U)$. From the choice of $V$ and the definition of $Q$, it is clear that $Uf^\sim(X) = \{V \in \mathcal{V} : V \subseteq f[X]\}$. Hence $Uf^\sim(X) = \mathcal{V}[f[X]]$ since $f[X]$ is $\mathcal{V}$ -substantial. From the definitions of $X$ and $Q$, it is clear that $Uf^\sim(X) = \{V \subseteq f[X] : f^\sim(V) \in \mathcal{U}\}$.

Hence $Uf^\sim(X) = f^\sim(U)f[X]$ since $f[X]$ is $f^\sim(U)$ -substantial, so that $U[f[X]] = f^\sim(U)f[X]$. But then $V = f^\sim(U)$ by substantialness. Therefore $f^\sim(U) \subseteq V$.

Step V: If $f^\sim(U) \subseteq V$, then $R(f(U)) \subseteq \mathcal{U}^*$. Pick $\{X, Q\} \in \mathcal{U}^*$. If $f^\sim(Q)$ isn't significant, then $R(f)(X, Q) = f^\sim(X), f^\sim(Q) \in \mathcal{U}^*$. If $f^\sim(Q)$ is significant, then so is $Q$, and for some $U \in \mathcal{U}$, $UUQ$ is $\mathcal{U}$ -substantial and $UX = \{U \in \mathcal{U} : \exists A \subseteq U \in \mathcal{Q}, U = UA\}$. But then $f^\sim(UUQ)$ is $f^\sim(U)$ -substantial and $f^\sim(U) \subseteq V$. To see that $f^\sim(X, Q) \in \mathcal{U}^*$, we need to show that $Uf^\sim(X) = \{V \subseteq f^\sim(U) : \exists A \subseteq U \in \mathcal{Q}, V = Uf^\sim(A)\}$.

Pick $V \in Uf^\sim(X)$; then for some $U \in \mathcal{U}$ and $A \subseteq U \in \mathcal{Q}$, $U = UA$ and $f[U] = V$. We have $f^\sim(f[U]) \subseteq \mathcal{U} \subseteq \mathcal{U}$. 

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since if not, there would be some \( Q_1 \in \mathcal{A} \) and \( Q_2 \in \mathcal{U} \) such that \( f(Q_1) \cap f(Q_2) \neq \emptyset \), in which case \( f^\sim(Q) \) wouldn't be significant. Consequently, \( f^\sim(f[U]) \in \mathcal{U} \) since \( 
abla U \mathcal{Q} \) is \( \mathcal{U} \)-substantial. Hence \( f[U] \in f^\sim_{\mathcal{U}}(\mathcal{U}) \). Conversely, if \( V \in f^\sim_{\mathcal{U}}(\mathcal{U}) \) and for some \( Q \subseteq \mathcal{U} \), \( V = Uf^\sim(Q) \), then \( f^\sim(V) \cap \nabla U \mathcal{Q} = U \mathcal{A} \) again since \( f^\sim(Q) \) would otherwise not be significant. Since \( f^\sim(V) \subseteq \mathcal{U} \) and \( \nabla U \mathcal{Q} \) is \( \mathcal{U} \)-substantial, \( f^\sim(V) \cap \nabla U \mathcal{Q} \subseteq \mathcal{U} \). Hence \( f^\sim(V) \cap \nabla U \mathcal{Q} \subseteq \nabla \nabla U \mathcal{Q} \), and

\[
\nabla \nabla U \mathcal{Q} = f[U] = V \in Uf^\sim(X) .
\]

Therefore \( f^\sim(f[X],Q_3) \in V^* \), as required.

We have just shown that the map \( \mathcal{U} \mapsto \nabla^* \) induces a realization of \( S((P^*)^2) \) in \( S(\mathcal{A}) \). Since for each structure \( \mathcal{U} \subseteq \mathcal{P}^{2}(X) \), \( \mathcal{U}^* \subseteq (P^*)^4(X) \), the same construction may be considered as a realization of \( S((P^*)^2) \) in \( S((P^*)^4) \). Using a similar construction, we can now show that \( (P^*)^5 \) majorizes \( (P^*)^2 \). For each set \( X \), each \( \mathcal{U} \subseteq \mathcal{P}(X) \), and each \( \mathcal{U} \)-substantial \( A \subseteq X \), let \( \mathcal{U}_A \) be the set of all \( \mathcal{A}, Q \in R(A) \) such that \( UUQ = A \) and if \( Q \) is significant, then \( UX = \{ U \in \mathcal{U} : \exists A \subseteq UQ \} \), \( \mathcal{U} = UU3 \). Define a functor \( E : \text{Set} \rightarrow \text{Set} \) as follows:

for each set \( X \), let \( E(X) = \mathcal{U}_A : \mathcal{U} \subseteq \mathcal{P}(X) \) and \( A \) is \( \mathcal{U} \)-substantial; for each function \( f : X \rightarrow Y \) and \( \mathcal{U}_A \in E(X) \), let \( E(f)(\mathcal{U}_A) = (P^*)^5(f) \). \( E \) is in fact a functor, as a result of the following

Step VI: For any given \( f : X \rightarrow Y \) and \( \mathcal{U}_A \in E(X) \),

\( E(f)(\mathcal{U}_A) = f^\sim_{\mathcal{U}}(\mathcal{U})_{f[A]} \). The argument of step V
shows that $E(\mathcal{C}) \subseteq \mathcal{C}$.

Now pick $\{V, S\} \subseteq \mathcal{C}$. Let $F = \{F, V, S\}$, and let $G = \{G, X, S\} : \{G, X, S\} \subseteq F$.

Clearly, $G \subseteq X, S$, and $U\mathcal{C} = U\mathcal{C} = A$, so that $\{X, S\} \subseteq \mathcal{C}$. If $\mathcal{C}$ is not significant, neither is $\mathcal{C}$, and thus $\{X, S\} \subseteq \mathcal{C}$. Assume $\mathcal{C}$ is significant; then so is $\mathcal{C}$. To see that $\{X, S\} \subseteq \mathcal{C}$, we need to show that $U\mathcal{C} = \{\mathcal{C} \subseteq \mathcal{C} : \mathcal{C} \subseteq \mathcal{C} \}$. First pick $U \subseteq \mathcal{C}$; then $F \subseteq \mathcal{C}$, so that for some $\mathcal{C} \subseteq \mathcal{C}$, $\mathcal{C} \subseteq \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{C} \subseteq \mathcal{C}$.

For each set $X$, let $\mathcal{C} \subseteq \mathcal{C}$ be the inclusion map from $E(\mathcal{C})$ to $\mathcal{C}$. $\mathcal{C}$ is clearly a mono transformation from $E$ to $\mathcal{C}$. Now define an epitransformation $\gamma$ from $E$ to $\mathcal{C}$ as follows: $\gamma \subseteq \mathcal{C}(X)$, $\mathcal{C}(X) = \mathcal{C}$. Each $\mathcal{C}$ is well-defined since each $\mathcal{C}$ contains a pair $\{\mathcal{C}, S\}$ such that $U\mathcal{C} = U\mathcal{C}$. (just let $\mathcal{C} = \{\mathcal{C}, S\} : \mathcal{C}, \mathcal{C} \subseteq \mathcal{C}, \mathcal{C} \subseteq \mathcal{C}$.)

Each $\mathcal{C}$ is clearly onto; to see that $\mathcal{C}$ is a natural transformation from $E$ to $\mathcal{C}$, pick $\mathcal{C}: X \rightarrow \mathcal{C}$ and $\mathcal{C} \subseteq \mathcal{C} \in E(X)$; then $\mathcal{C}: (\mathcal{C}) = \mathcal{C} = \mathcal{C} = \mathcal{C} = \mathcal{C}$.
Therefore \((P^+)^5\) majorizes \((P^-)^2\).

2 Theorem. If \(G_1, \ldots, G_n\) are constructively majorizable functors and \(\Delta_1, \ldots, \Delta_n\) are types, then \(S((G_1, \Delta_1), \ldots, (G_n, \Delta_n))\) is realizable in \(S((P^+)^k \circ V_M)\) for some set \(M\) and natural number \(k\).

Proof. The numbered theorems which will be referred to are those of [1]. By Theorem 6.5, \(S((G_1, \Delta_1), \ldots, (G_n, \Delta_n))\) is realizable in \(S((P^-)^k \circ V_M)\) for some number \(k\) and set \(M\). If \(k\) is odd, then \(S((P^-)^k \circ V_M)\) is realizable in \(S((P^-)^{k+1} \circ V_M)\) by Theorem 1.5. Hence \(S((G_1, \Delta_1), \ldots, (G_n, \Delta_n))\) is realizable in some \(S((P^-)^{2m} \circ V_M)\) by Corollary 3.7 and the above lemma, \((P^-)^{2m} \circ V_M\) is majorized by \((P^+)^{5m} \circ V_M\). Hence by Theorem 6.1, \(S((P^-)^{2m} \circ V_M)\) is realizable in \(S((P^+)^{5m} \circ V_M)\).

Problem: Characterize the class of all categories \(S(P)\) which can be realized in some \(S((P^+)^k \circ V_M)\) (or, equivalently, \(S((P^-)^k \circ V_M)\)). Characterize the class of all categories \(S(P, \Delta)\) which can be realized in some \(S((P^+)^k, \Gamma)\) (equivalently, in \(S((P^-)^k, \Gamma)\)).

The above theorem may be extended to the infinite case with the help of the following result.

3 Lemma. For each monotransformation \(\tau : I \to (P^+)^m\) there is an \(m \geq n\) and a monotransformation \(\Theta : (P^+)^m \to (P^+)^m\) such that \(\Theta \tau = \xi^m\), where \(\xi : I \to P^+\) is the unique monotransformation.
Proof: First we need some facts about natural transformations from $I$ to $(P^+)^n$. By Remark 2.9 of [2], the natural transformations from $I$ to $(P^+)^n$ are in 1-1 correspondence with the elements of $(P^+)^n(\emptyset)$, and for any set $A \in (P^+)^n(\emptyset)$, we may let $\tau_{m,A}$ be the transformation such that for each set $X$ and $x \in X$, $\tau_{m,A}(x) = (P^+)^n(\varepsilon_x)(A)$, where $\varepsilon_x: \emptyset \to X$ is given by $\varepsilon_x(\emptyset) = x$. Since $\tau_{m,A}$ doesn't depend on $X$ in a significant way, we will usually drop this third subscript. Notice that

if $A \in (P^+)^n(\emptyset)$, then

$$
\tau_{m+1,A}(x) = (P^+)^{n+1}(\varepsilon_x)(A) = ((P^+)^n(\varepsilon_x)(a); a \in A) = \tau_{m,A}(a); a \in A.
$$

1) The following are equivalent:

a) $\tau_{m,A}$ is a monomorphism

b) $\text{rank } A = m$ (where $\text{rank } A$ is inductively defined as the smallest ordinal greater than $\text{rank } a$, for all $a \in A$).

c) $\forall x, U^n \tau_{m,A}(x) = x$, where for any set $S$, $U^0 S = S$ and $U^{n+1}(S) = \{ U^n b; b \in S \}$.

d) $\exists x, U^n \tau_{m,A}(x) \neq \emptyset$.

Proof: The only element of $(P^+)^0(\emptyset)$ is $\emptyset$, and so $\tau_{0,\emptyset} : I \to I$ is the identity transformation; $\tau_{0,\emptyset}$ clearly satisfies the four conditions. By induction, assume for $m \geq 0$ that the four conditions are equivalent. Pick $A \in (P^+)^{n+1}(\emptyset)$. Then $\text{rank } A = m + 1$ iff for some $a \in A$, $\text{rank } a = m$, in which case $\tau_{m,A}$ would satisfy the four conditions. Thus if $\text{rank } A = m + 1$, then

$$
U^{n+1} \tau_{m+1,A}(x) = U^{n+1} \tau_{m,A}(x); a \in A
$$

$$
= U^0 U^n \tau_{m,A}(x); a \in A
$$

$$
\begin{cases}
- x, & \text{if } 3a \in A, \text{ rank } a < m \\
- U^0 x, & \text{if } \forall a \in A, \text{ rank } a = m
\end{cases}
$$

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and so the four conditions hold. But if \( \text{rank} \ A < m+1 \), then
\[
U^{n+1}_{m+1,A}(x) = U \cup U^m_{m,A}(x); a \in A \cap U \cap \emptyset = \emptyset,
\]
and they don't hold.

For any set \( X \), let \( \sigma_X \) be the unique map from \( X \) to \( \{\emptyset\} \). For each natural number \( k \) and \( C \in (P^+)^k(X) \), define the \( k \)-type of \( C \) to be \((P^+)^k(\sigma_X)(C)\). Notice that a set \( A \in (P^+)^{k+1}(\{\emptyset\}) \) is the \( k+1 \)-type of \( C \in (P^+)^{k+1}(X) \) iff \( A \) is the set of \( k \)-types of elements of \( C \). We will need the following properties of natural transformations from \((P^+)^\hat{\delta} \) to \((P^+)^k \):

2) Suppose that \( A \in (P^+)^{k}(\{\emptyset\}) \) and \( \text{rank} \ A < k \). Then for any set \( Y \), \( A \in (P^+)^k(Y) \), as can be easily seen by induction on the rank of \( A \). Consequently the constant transformation \( \gamma \) from \((P^+)^\hat{\delta} \) to \((P^+)^k \), given by \( \forall X, \forall C \in (P^+)^\hat{\delta}(X), \gamma_X(C) = A \) is natural.

3) If \( C \in (P^+)^\hat{\delta}(X) \) and \( f : X \rightarrow Y \), then \((P^+)^\hat{\delta}(C) \) has the same \( \hat{\delta} \)-type as \( C \) since
\[
(P^+)^\hat{\delta}(\sigma_Y)((P^+)^\hat{\delta}(f)(C)) = (P^+)^\hat{\delta}(\sigma_Y f)(C)
= (P^+)^\hat{\delta}(\sigma_X(C)).
\]
From this fact, it follows immediately that given \( \phi, \psi : (P^+)^\hat{\delta} \rightarrow (P^+)^k \) and \( \Delta \subseteq (P^+)^{k}(\{\emptyset\}) \), one can define a natural transformation \( \Theta : (P^+)^\hat{\delta} \rightarrow (P^+)^k \) by \( \forall X, \forall C \in (P^+)^\hat{\delta}(X), \)
\[
\Theta_X(C) = \begin{cases} 
\phi_X(C), & \text{if the } \hat{\delta} \text{-type of } C \text{ is in } \Delta \\
\psi_X(C), & \text{otherwise.}
\end{cases}
\]
4) The same fact guarantees that if for each \( a \in A \), we
5) Given natural transformations \( \varphi_1, \ldots, \varphi_p \) from \((P^+)\hat{\delta} \) to \((P^+)\hat{\kappa} \), we can define a product transformation \( \varphi_1 \times \ldots \times \varphi_p : (P^+)\hat{\delta} \to (P^+)\hat{\kappa} \) as follows: inductively define \( \langle x \rangle = \{ x \} \), and

\[
\langle x_1, \ldots, x_{m+1} \rangle = \langle x_1, \ldots, x_m \rangle \cup \varphi_m(x_{m+1}).
\]

It is easy to see that \( \cap \langle x_1, \ldots, x_{m+1} \rangle = \langle x_1, \ldots, x_m \rangle \) and (by induction) that \( \cup^m \langle x_1, \ldots, x_{m+1} \rangle = \{ x_1, \ldots, x_{m+1} \} \), so that this is an acceptable convention for \( m \)-tuples. Also, if \( x_1, \ldots, x_p \in X \), then \( \langle x_1, \ldots, x_p \rangle \in (P^+)\kappa(X) \); hence if \( C \in (P^+)\kappa(X) \), then \( \varphi(C) = (\varphi_1 \times \ldots \times \varphi_p)(C) \in (P^+)\kappa+\kappa(X) \). Notice that if \( \langle D_1, \ldots, D_p \rangle \) are of \( \kappa \)-type \( \varphi(X, \emptyset) \), then \( \langle D_1, \ldots, D_p \rangle \) is of \( \kappa + \kappa \)-type \( \varphi(X, \emptyset) \).

We can now find the required \( \theta : (P^+)^m \to (P^+)^m \) as follows: for \( m = 0 \) the only monotransformation from \( I \) to \((P^+)^m \) is the identity. For \( m = 1 \), the only one is \( \pi \) itself. In either case we may let \( \theta \) be the identity on \((P^+)^m \). Notice that if \( a \in (P^+)^m(\emptyset) \), then for each set \( X \) and \( x \in X \), \( \tau_{m, a} \) is characterized by the fact that the \( m \)-type of \( \tau_{m, a}(x) \) is \( a \), since

\[
(P^+)^m(\pi_X(\tau_{m, a}(x))) = \tau_{m, a}(\pi_X(x)) = \tau_{m, a}(a) = a.
\]

Our inductive assumption will, accordingly, be that for
\( m \geq 1 \), there is a \( \mathfrak{L} \geq m \) such that for each monomorphism \( \varphi : (P^+)^m \rightarrow (P^+)^m \), there is a monomorphism \( \theta : (P^+)^m \rightarrow (P^+)^m \) such that whenever \( C \in (P^+)^m \) is of \( m \)-type \( \mathfrak{a} \), \( \varphi_g(C) \) is of \( \mathfrak{L} \)-type \( \xi^\mathfrak{L}(\emptyset) \). We then have, in particular that \( \forall x, \varphi_{m,a}(x) \) is of \( m \)-type \( \mathfrak{a} \), and \( \varphi_g \varphi_{m,a}(x) \) is of \( \mathfrak{L} \)-type \( \xi^\mathfrak{L}(\emptyset) \), so that \( \varphi_g \varphi_{m,a} = \varphi_{\mathfrak{L},\mathfrak{L}}(\emptyset) = \xi^\mathfrak{L} \). Let \( \varphi_{m+1,A} : (P^+)^m \rightarrow (P^+)^{m+1} \) be any fixed monomorphism. Let \( A = \{a_1, \ldots, a_n\} \) be any indexing of \( A \) such that \( a_1, \ldots, a_n \) are the elements of \( A \) of rank \( m \). For each \( a_i \), let \( \theta_i \) be a monomorphism from \( (P^+)^m \) to \( (P^+)^m \) satisfying the induction hypothesis. Define \( \varphi_i : (P^+)^{m+1} \rightarrow (P^+)^{m+1} \) by \( \forall x, \forall C \in (P^+)^{m+1}(X) \):

\[
\varphi_i(C) = \{ \varphi_i(C) : C \in \mathcal{C} \quad \text{and} \quad \mathcal{C} \quad \text{is of m-type} \quad a_i \}.
\]

Let \( \theta : (P^+)^{m+1} \rightarrow (P^+)^{m+1} \) be given by \( \forall C \in (P^+)^{m+1}(X) \),

\[
\theta(X) = \varphi_1 \times \ldots \times \varphi_m(C), \quad \text{if} \quad C \quad \text{is of m+1-type} \quad \mathfrak{a},
\]

and

\[
\theta(X) = \begin{cases} 
\xi^{m+\mathfrak{a}+1}(\emptyset), & \text{if} \quad \emptyset \quad \text{is of m+1-type} \quad \mathfrak{a} \quad \text{and} \quad \emptyset \quad \text{by (4), and} \quad \emptyset \quad \text{is natural by (3), (5), and (4) and (2).}
\end{cases}
\]

To see that if \( C \) is of \( m+1 \)-type \( \mathfrak{a} \), then \( \theta(X) \) is of \( m+\mathfrak{L}+1 \)-type \( \xi^{m+\mathfrak{L}+1}(\emptyset) \), notice first that \( \{a_1, \ldots, a_n\} \) is nonempty by (1) since \( \varphi_{m,A} \) is a monomorphism. Each element of each \( \varphi_{i+m}(C) \) is of \( \mathfrak{L} \)-type \( \xi^\mathfrak{L}(\emptyset) \) by the inductive assumption. Hence each element of \( \varphi_1 \times \ldots \times \varphi_m(C) \) is of \( \mathfrak{L} \)-type \( \xi^\mathfrak{L}(\emptyset) \), so that \( \varphi_1 \times \ldots \times \varphi_m(C) \) is of \( \mathfrak{L} \)-type \( \xi^{m+\mathfrak{L}+1}(\emptyset) \).

Finally, each \( \theta(X) \) is mono: let \( \theta(X) \) be given.
\( \mathcal{C} \) may be recovered as follows: if \( \emptyset \in \theta_X(\mathcal{C}) \), then 
\( \mathcal{C} = U^{m+\lambda-\lambda} \theta_X(\mathcal{C}) \). Assume \( \emptyset \notin \theta_X(\mathcal{C}) \). Then \( \mathcal{C} \) is of \( m+1 \) -type \( A \). Let \( \mathcal{C} = \mathcal{C}_0 \cup \mathcal{C}_1 \), where \( \mathcal{C}_1 \) is the set of elements of \( \mathcal{C} \) of rank less than \( m \), and \( \mathcal{C}_0 \) is the rest. We know that \( (P^+)^{m+1}(\pi_X)(\mathcal{C}) = A = \{ a_1, \ldots, a_m \} \cup \{ e_1, \ldots, e_2 \} \).

By an easy induction we have that \( YC \in (P^+)^m(X) \), \( \text{rank } C \geq n \) iff \( \text{rank } (P^+)^m(\pi_X)(C) = m \), and that if \( \text{rank } C < m \), then \( (P^+)^m(\pi_X)(C) = C \). Consequently, \( \mathcal{C}_1 = \{ e_1, \ldots, e_2 \} \), and \( \{ a_1, \ldots, a_m \} \) is the \( m+1 \)-type of \( \mathcal{C}_0 \).

For each \( a_i \), let \( \mathcal{N}_i \) be a left inverse function for \( \theta_{iX} \); clearly,
\[
\mathcal{C}_0 = \{ \mathcal{N}_i(D) : D \text{ is the } i \text{th element of some } \mu \text{-tuple in } \theta_X(\mathcal{C}) \}.
\]

As it stands, the number \( m_\lambda = \lambda + \mu + 1 \) depends on \( A \), since \( \mu \) does. However, a uniform \( m = \max \{ m_\lambda : A \in (P^+)^{m+1}(X) \} \) is easily obtained by composing \( \theta \) with \( \xi^{m-m_\lambda} \). This completes the induction.

**4 Theorem.** Let \( F_\lambda (\lambda \in \Gamma) \) be TB-functors (in the sense of [21]), and \( A_\lambda (\lambda \in \Gamma) \) types. Then there is an ordinal \( \alpha \) and a set \( A \) such that
\[
S((F_\lambda, A)_{\lambda \in \Gamma}) \Rightarrow S((P^+)_{\alpha} \cdot V_A).
\]

**Proof.** Let \( A : I \rightarrow (P^+)^2 \) be the monomorphism given by \( YX, \forall x \in X, A_X(x) = \{ A = X : x \in A \} \). Define \( \mu : I \rightarrow E \) by \( YX, \forall x \in X, \mu_X(x) = A_X(x)_{\{x\}} = \{ \{ \emptyset, \mathcal{Q} \in eX(\{x\}) : UU \mathcal{Q} = \{ x \} \}, \text{ and if } \mathcal{Q} \text{ is significant, then } UX = \{ \{ x \} \} \). The condition that \( UU X \leq UU \mathcal{Q} = \{ x \} \)
forces $\mu_X(x)$ to be independent of $X$, and a moment's thought shows that $\mu$ is a monomorphism. As at the end of Lemma 1, let $\varphi: E \to (P^+)^\beta$ be the monomorphism given by the equation $\varphi_X(\mathcal{U}_A) = \mathcal{U}_A$, and let $\psi: E \to (P^-)^2$ be the epimorphism given by $\psi_X(\mathcal{U}_A) = \mathcal{U}$.

Then $\psi\mu = \lambda$. Finally, for some $m$ bigger than $5$, we may let $\theta: (P^+)^5 \to (P^+)^m$ be a monomorphism such that $\theta\psi\mu = \xi^m$.

We need to show that any functor of the form $((P^-)^2)^\beta$ is majorized by some $(P^+, \xi^\alpha)$. Let $\alpha$ be a limit ordinal larger than $\beta$. Then $((P^-)^2)^\beta < ((P^-)^2)^\alpha$ by Lemma 3.7 of [2]. The equations $\psi\mu = \lambda$ and $\theta\psi\mu = \xi^m$, and Lemma 2.8 of [2] show that

\[( (P^-)^2, \lambda)^\alpha \prec (E, \mu)^\alpha \prec (P^+, \xi)^\alpha \prec (P^+)^m, \xi^m)^\alpha.\]

But by Lemma 2.4 of [2], $(P^+, \xi^m)^\alpha \prec (P^+, \xi)^\alpha$, since the first colimit is just being taken over a subsequence of the second. Now by Theorem 3.7 of [1], we have $((P^-)^2, \lambda)^\alpha \circ \mathcal{V}_A \prec (P^+, \xi)^\alpha \circ \mathcal{V}_A$, for any set $A$, and thus by Theorem 6.1 of [1], $S((P^-)^2, \lambda)^\alpha \circ \mathcal{V}_A \Rightarrow S((P^+, \xi)^\alpha \circ \mathcal{V}_A)$.

Finally, let $S(F_L, \Delta_L)_{L \in \mathcal{P}}$ be as in the statement of the theorem. Then by Theorem 4.2 of [2], $S((P^-)^2, \lambda)^\beta \circ \mathcal{V}_A \Rightarrow S((P^-)^2, \lambda)^\alpha \circ \mathcal{V}_A$, for some ordinal $\beta$ and set $A$ and the theorem follows.

References

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