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NOTES ON RADICAL FILTERS OF IDEALS

Tomáš KEPKA, Praha

Abstract: Let  $R$  be a ring and  $\mathcal{M}$  be a non-empty set of left ideals of  $R$ . Denote by  $\mathcal{F}(\mathcal{M})$  the radical filter generated by  $\mathcal{M}$ . In this paper we give a certain characterization of  $\mathcal{F}(\mathcal{M})$ .

Key words: Radical filter, hereditary torsion class, hereditary radical.

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In the following,  $R$  will be an associative ring with unit and the word "module" means a unitary left  $R$ -module. Further, we shall denote by  $R\text{-mod}$  the category of all the  $R$ -modules and  $\mathcal{F}(M)$  will be the set of all submodules in  $M$  for any  $M \in R\text{-mod}$ . Let  $\mathcal{M} \subseteq \mathcal{F}(R)$  be a non-empty subset. Consider the following six conditions for  $\mathcal{M}$ .

- (F<sub>1</sub>) If  $I \in \mathcal{M}$ ,  $K \in \mathcal{F}(R)$  and  $I \subseteq K$ , then  $K \in \mathcal{M}$ .
- (F<sub>2</sub>) If  $I \in \mathcal{M}$  and  $\lambda \in R$ , then  $(I : \lambda) = \{ \rho \in R, \rho \lambda \in I \} \in \mathcal{M}$ .
- (F<sub>3</sub>) If  $I, K \in \mathcal{M}$ , then  $I \cap K \in \mathcal{M}$ .
- (F<sub>4</sub>) If  $I, K \in \mathcal{M}$ , then  $I \cdot K \in \mathcal{M}$  and
- (F<sub>5</sub>) If  $I \in \mathcal{M}$ ,  $K \in \mathcal{F}(R)$ ,  $K \subseteq I$  and  $(K : \lambda) \in \mathcal{M} \forall \lambda \in I$ , then  $K \in \mathcal{M}$ .
- (F<sub>6</sub>) If  $I \in \mathcal{M}$ ,  $K \in \mathcal{F}(R)$  and  $(K : \lambda) \in \mathcal{M} \forall \lambda \in I$ , then  $K \in \mathcal{M}$ .

The set  $\mathcal{M}$  is called a filter (a radical filter) if it satisfies the conditions  $(F_1)$ ,  $(F_2)$ ,  $(F_3)$  ( $(F_1)$ ,  $(F_2)$ ,  $(F_5)$ ). As it is easy to show, any radical filter satisfies all the six conditions  $(F_1) \dots (F_6)$ . Recall that there is a one-to-one correspondence between radical filters and so called hereditary radicals. A hereditary radical is an arbitrary subfunctor of the identity  $\kappa$  having the following properties:

$$(i) \quad \kappa \left( \frac{M}{\kappa(M)} \right) = 0 \quad \forall M \in \mathcal{R}\text{-mod} ,$$

$$(ii) \quad \kappa(N) = N \cap \kappa(M) \quad \forall M \in \mathcal{R}\text{-mod} \quad \forall N \in \mathcal{S}(M) .$$

If  $\mathcal{M}$  is a radical filter, then the subfunctor  $\kappa$ , given by  $\kappa(M) = \{m \mid (0 : m) \in \mathcal{M}\}$ , is a hereditary radical. Conversely, if  $\kappa$  is a hereditary radical then  $\{I \mid I \in \mathcal{S}(\mathcal{R}), \kappa(\mathcal{R}/I) = \mathcal{R}/I\}$  is a radical filter. (For the proof see e.g. [4].) A non-empty class of modules  $\mathcal{M}$  is said to be a hereditary torsion class, if it is closed under submodules, homomorphic images, extensions and direct sums. In this case, the subfunctor  $\kappa$ ,  $\kappa(M) = \sum_{N \in \mathcal{S}(M) \cap \mathcal{M}} N$  is a hereditary radical. Conversely, if  $\kappa$  is a hereditary radical then  $\{M \mid \kappa(M) = M\}$  is a hereditary torsion class. Since the intersection of any set of radical filters is a radical filter, we can consider the complete lattice  $\mathcal{L}(\mathcal{R})$  of all radical filters of the ring  $\mathcal{R}$ . Finally, denote by  $\mathcal{K}(\mathcal{R})$  the set of all the subsets  $\mathcal{M} \subseteq \mathcal{S}(\mathcal{R})$  which satisfy the conditions  $(F_1)$  and  $(F_2)$ . It is obvious that  $\mathcal{K}(\mathcal{R})$  is a sublattice in the lattice

$2^{\mathcal{G}(R)}$  of all subsets of  $\mathcal{G}(R)$  .

2. If  $M \in R\text{-mod}$  and  $K \in \mathcal{G}(M)$  then we denote by  $\mathcal{E}^1(K, M)$  the set  $\{N \mid N \in \mathcal{G}(M), K \subseteq N\}$  and by  $\mathcal{E}^2(K, M)$  the set  $\{N \mid N \in \mathcal{G}(M), K \subseteq N \text{ and } N/K \text{ is essential in } M/K\}$  . Further,  $\mathcal{E}^3(K, M)$  will be  $\mathcal{E}^2(K, M) \cup \{K\}$  .

2.1. Lemma. Let  $M \in R\text{-mod}$  and  $K, L, N \in \mathcal{G}(M)$  be such that  $K \subseteq L \subseteq N$  . Then:

- (i)  $N \in \mathcal{E}^2(K, M)$  iff  $N \cap X = K$  implies  $X = K$  for arbitrary  $X \in \mathcal{G}(M)$  .
- (ii)  $N \in \mathcal{E}^2(L, M)$  implies  $N \in \mathcal{E}^2(K, M)$  .
- (iii)  $L \in \mathcal{E}^2(K, M)$  implies  $N \in \mathcal{E}^2(K, M)$  .

Proof. Obvious.

Before we proceed further, let us introduce the following notation. If  $M \in R\text{-mod}$  and  $\emptyset \neq \mathcal{M} \subseteq \mathcal{G}(M)$  , then by  $\kappa_{\mathcal{M}}$  we shall mean the hereditary radical corresponding to the hereditary torsion class, which is generated by all the factor-modules  $M/N$  ,  $N \in \mathcal{M}$  . Further put  $\mathcal{A}(\mathcal{M}) = \{S \mid S \in \mathcal{G}(M), \exists m \in M \setminus S$   
 $\forall n \in M \forall \lambda \in R \setminus (S : m) \forall N \in \mathcal{M} \exists \sigma \in (N : m) \text{ such that } \sigma \lambda m \notin S\}$  and  $\mathcal{B}(\mathcal{M}) = \mathcal{G}(M) \setminus \mathcal{A}(\mathcal{M})$  . Thus  $\mathcal{B}(\mathcal{M}) = \{S \mid S \in \mathcal{G}(M), \forall m \in M \setminus S \exists n \in M \exists \lambda \in R \setminus (S : m) \exists N \in \mathcal{M}$   
such that  $(N : m) \subseteq (S : \lambda m)\}$  .

2.2. Lemma. Let  $M \in R\text{-mod}$ ,  $A \in \mathcal{G}(M)$  and

$\emptyset \neq \mathcal{M} \subseteq \mathcal{G}(M)$ . Then  $\lambda \in \mathcal{A}(M)$  iff there is  $m \in M \setminus A$  such that

$$\text{Hom}_{\mathbb{R}}(\mathbb{B}/N, \mathbb{R}m+A/A) = 0$$

for all  $N \in \mathcal{M}$  and  $B \in \mathcal{E}^1(N, M)$ .

Proof. (i) Let  $\lambda \in \mathcal{A}(M)$ . Then there is  $m \in M \setminus A$  such that  $(N: n) \not\subseteq (A: \lambda m)$  for any  $n \in M$ ,  $N \in \mathcal{M}$  and  $\lambda \in \mathbb{R} \setminus (A: m)$ . If  $\varphi: \mathbb{B}/N \rightarrow \mathbb{R}m+A/A$  is non-zero, then  $\varphi(\mathcal{L} + N) = \varphi m + A \neq 0$  for some  $\mathcal{L} \in \mathbb{B}$  and  $\varphi \in \mathbb{R}$ . Hence  $\varphi \in \mathbb{R} \setminus (A: m)$  and  $(N: \mathcal{L}) \subseteq (A: \varphi m)$ , a contradiction.

(ii) Let  $A$  satisfy the condition of the lemma. If  $(N: m) \subseteq (A: \lambda m)$  for some  $N \in \mathcal{M}$  and  $\lambda \in \mathbb{R} \setminus (A: m)$ , then the mapping  $\varphi: \mathbb{R}m+N/N \rightarrow \mathbb{R}m+A/A$  defined by

$\varphi(\varphi m + N) = \varphi \lambda m + A \quad \forall \varphi \in \mathbb{R}$ , is a non-zero homomorphism, a contradiction.

2.3. Lemma. Let  $M \in \mathbb{R}\text{-mod}$ ,  $K \in \mathcal{G}(M)$  and  $\emptyset \neq \mathcal{M} \subseteq \mathcal{G}(M)$  be such that  $K \in \mathcal{B}(M)$ . Then:

(i)  $S \in \mathcal{E}^2(K, M)$ , where  $S/K = \kappa_m(M/K)$ .

(ii)  $\kappa_m(M/K) \neq 0$ , provided  $M \neq K$ .

Proof. (i) Let  $m \in M \setminus K$  be arbitrary. In view of Lemma 2.2, there is  $N \in \mathcal{M}$  and  $B \in \mathcal{G}(M)$  such that

$N \subseteq B$  and  $\text{Hom}_{\mathbb{R}}(\mathbb{B}/N, \mathbb{R}m+K/K) \neq 0$ . Since

$\kappa_m(\mathbb{B}/N) = \mathbb{B}/N, \kappa_m(\mathbb{R}m+K/K) \neq 0$ . However,

$\kappa_m(\mathbb{R}m+K/K) = \mathbb{R}m+K/K \cap \mathbb{S}/K$ , and consequently  $\mathbb{S}/K$  is essential in  $\mathbb{M}/K$ .

(i) There is  $m \in M \setminus K$ , and hence (by Lemma 2.2)

$\text{Hom}_{\mathbb{R}}(\mathbb{B}/N, \mathbb{R}m+K/K) \neq 0$  for some  $N \in \mathcal{M}$  and  $\mathbb{B} \in \mathcal{E}^1(N, M)$ . Thus  $0 \neq \kappa_m(\mathbb{R}m+K/K) \subseteq \kappa_m(\mathbb{M}/K)$ .

2.4. Lemma. Let  $M \in \mathbb{R}\text{-mod}$ ,  $K \in \mathcal{F}(M)$  and  $\emptyset \neq \mathcal{M} \subseteq \mathcal{F}(M)$ . Then the following are equivalent:

- (i)  $\mathcal{E}^3(K, M) \cap \mathcal{Q}(M) \neq \emptyset$ .
- (ii)  $\mathcal{E}^1(K, M) \cap \mathcal{Q}(M) \neq \emptyset$ .
- (iii) There are  $A \in \mathcal{E}^3(K, M)$  and  $S \in \mathcal{F}(M)$  such that  $A \not\subseteq S$  and  $\kappa_m(S/A) = 0$ .
- (iv) There are  $A \in \mathcal{E}^1(K, M)$  and  $S \in \mathcal{F}(M)$  such that  $A \not\subseteq S$  and  $\kappa_m(S/A) = 0$ .
- (v)  $\kappa_m(\mathbb{M}/K) \neq \mathbb{M}/K$ .

Proof. (i) implies (ii) and (iii) implies (iv) trivially. (i) implies (iii). Let  $A \in \mathcal{E}^3(K, M) \cap \mathcal{Q}(M)$ . By Lemma 2.2, there is  $m \in M \setminus A$  such that

$\text{Hom}_{\mathbb{R}}(\mathbb{B}/N, \mathbb{R}m+A/A) = 0$  for all  $N \in \mathcal{M}$  and  $\mathbb{B} \in \mathcal{E}^1(N, M)$ . From this, one can easily derive

$\kappa_m(\mathbb{R}m+A/A) = 0$ . Now it is sufficient to put  $S = \mathbb{R}m+A/A$ .

Similarly we can prove (ii) implies (iv).

(iv) implies (v). If  $\kappa_m(M/K) = M/K$ , then  $\kappa_m(S/A) = S/A$  for all  $A, S \in \mathcal{E}^1(K, M)$  such that  $A \subseteq S$ .

(v) implies (i). Assume, on the contrary, that  $K \in \mathcal{B}(M)$ , and therefore, in view of Lemma 2,3,  $S \in \mathcal{E}^2(K, M)$ , where

$S/K = \kappa_m(M/K)$ . Using Lemma 2.3 again, we get

$\kappa_m(M/S) \neq 0$ , a contradiction.

2.5. Theorem. Let  $\mathcal{M} \subseteq \mathcal{F}(R)$  be a non-empty subset. Then  $\mathcal{F}(\mathcal{M}) = \{I \mid I \in \mathcal{F}(R), \mathcal{E}^1(I, R) \subseteq \mathcal{B}(\mathcal{M})\} = \{I \mid I \in \mathcal{F}(R), \mathcal{E}^3(I, R) \subseteq \mathcal{B}(\mathcal{M})\}$ .

Proof. The theorem follows from Lemma 2.4, since  $\mathcal{F}(\mathcal{M}) = \{I \mid I \in \mathcal{F}(R), \kappa_m(R/I) = R/I\}$ .

2.6. Corollary. A non-empty subset  $\mathcal{R} \subseteq \mathcal{F}(R)$  is a radical filter iff it satisfies the following condition:

(F<sub>4</sub>) If  $I \in \mathcal{F}(R)$  and  $\forall K \in \mathcal{E}^1(I, R) \forall \kappa \in R \setminus K \exists \lambda \in R \exists \lambda \in R \setminus (K : \kappa) \exists L \in \mathcal{R}$  such that  $(L : \lambda) \subseteq (K : \lambda \kappa)$ , then  $I \in \mathcal{R}$ .

Proof. This corollary is only a transcription of Theorem 2.5.

For a non-empty subset  $\mathcal{M} \subseteq \mathcal{F}(R)$  put  $\mathcal{C}(\mathcal{M}) = \{I \mid \exists \lambda \in R \exists K \in \mathcal{M}$  such that  $(K : \lambda) \subseteq I\}$  and  $\mathcal{D}(\mathcal{M}) = \{I \mid \forall \lambda \in R \setminus I \exists \varphi \in R \setminus (I : \lambda)$  such that  $(I : \varphi \lambda) \in \mathcal{M}\}$ .

2.7. Corollary. Let  $\mathcal{M} \subseteq \mathcal{F}(\mathbb{R})$  be a non-empty subset. Then  $\mathcal{F}(\mathcal{M}) = \{I \mid I \in \mathcal{F}(\mathbb{R}), \mathcal{E}^1(I, \mathbb{R}) \subseteq \mathcal{D}(\mathcal{C}(\mathcal{M}))\} = \{I \mid I \in \mathcal{F}(\mathbb{R}), \mathcal{E}^3(I, \mathbb{R}) \subseteq \mathcal{D}(\mathcal{C}(\mathcal{M}))\}$ .

In particular, if  $\mathcal{M}$  satisfies  $(F_1)$  and  $(F_2)$ , then  $\mathcal{F}(\mathcal{M}) = \{I \mid \mathcal{E}^1(I, \mathbb{R}) \subseteq \mathcal{D}(\mathcal{M})\} = \{I \mid \mathcal{E}^3(I, \mathbb{R}) \subseteq \mathcal{D}(\mathcal{M})\}$ .

Proof. The corollary follows from Theorem 2.5, since  $\mathcal{B}(\mathcal{M}) = \mathcal{D}(\mathcal{C}(\mathcal{M}))$ , as one may check easily.

As a very easy consequence of 2.7 and 2.1 we get the following well-known result (see [3]).

2.8. Corollary. Let  $\mathcal{M} \subseteq \mathcal{F}(\mathbb{R})$  be a non-empty subset satisfying  $(F_1)$ ,  $(F_2)$  and let  $\mathcal{E}^2(0, \mathbb{R}) \subseteq \mathcal{M}$ . Then  $\mathcal{F}(\mathcal{M}) = \mathcal{D}(\mathcal{M})$ .

2.9. Corollary. Let  $\mathcal{M} \subseteq \mathcal{F}(\mathbb{R})$  be a non-empty subset and let  $\mathcal{H}(\mathcal{M}) = \{I \mid I \in \mathcal{F}(\mathbb{R}), \exists \lambda \in \mathbb{R} \setminus I \exists N \in \mathcal{M} \exists n \in \mathbb{R}$  such that  $(N : n) \subseteq (I : \lambda)\}$ . Then  $\mathcal{F}(\mathcal{M}) = \{I \mid \mathcal{E}^1(I, \mathbb{R}) \setminus \{I\} \subseteq \mathcal{H}(\mathcal{M})\}$ .

Proof. (i) Let  $I \in \mathcal{F}(\mathcal{M})$ ,  $I \neq \mathbb{R}$ . Then, by 2.6 (for  $\kappa = 1$ ), there are  $n \in \mathbb{R}$ ,  $\lambda \in \mathbb{R} \setminus (I : 1) = \mathbb{R} \setminus I$  and  $N \in \mathcal{M}$  with  $(N : n) \subseteq (I : \lambda)$ .

(ii) Let  $I \in \mathcal{F}(\mathbb{R})$  and  $\{I\} \subseteq \mathcal{H}(\mathcal{M})$ .

Set  $\mathcal{S}/I = \kappa_m(\mathbb{R}/I)$ . If  $\mathcal{S} = \mathbb{R}$ , then obviously  $I \in \mathcal{F}(\mathcal{M})$ . Suppose  $\mathcal{S} \neq \mathbb{R}$ . By the hypothesis, there are  $\lambda \in \mathbb{R} \setminus \mathcal{S}$ ,  $N \in \mathcal{M}$  and  $n \in \mathbb{R}$  such that  $(N : n) \subseteq (\mathcal{S} : \lambda)$ . Thus  $(\mathcal{S} : \lambda) \in \mathcal{F}(\mathcal{M})$  and  $\lambda + \mathcal{S} \in \kappa_m(\mathbb{R}/\mathcal{S})$ , a contradiction since  $\kappa_m(\mathbb{R}/\mathcal{S}) = 0$ .



2.10. Corollary. Let  $I \in \mathcal{F}(\mathbb{R})$  be a two-sided ideal;  $\varphi: \mathbb{R} \rightarrow \mathbb{R}/I$  be the canonical epimorphism and  $\mathcal{R} \subseteq \mathcal{F}(\mathbb{R}/I)$  be a radical filter. Put  $\mathcal{X} = \{K \mid K \in \mathcal{F}(\mathbb{R}), I \subseteq K \text{ and } \varphi(K) \in \mathcal{R}\}$ . Then  $\varphi(L) \in \mathcal{R}$  for all  $L \in \mathcal{F}(\mathcal{X})$ .

Proof. Let  $L \in \mathcal{F}(\mathcal{X})$  be arbitrary and  $K \in \mathcal{F}(\mathbb{R}) \setminus \{I\}$  be such that  $I \subseteq K$  and  $\varphi(L) \subseteq \varphi(K)$ . By 2.9, there are  $N \in \mathcal{X}$ ,  $\kappa \in \mathbb{R}$  and  $\sigma \in \mathbb{R} \setminus K$  with  $(N: \kappa) \subseteq (K: \sigma)$ . Since  $I$  is a two-sided ideal,  $I \subseteq (N: \kappa)$  and  $I \subseteq (K: \sigma)$ . Hence  $\varphi((N: \kappa)) = (\varphi(N): \varphi(\kappa)) \subseteq \varphi((K: \sigma)) = (\varphi(K): \varphi(\sigma))$ . However,  $\varphi(N) \in \mathcal{R}$  and  $\varphi(\sigma) \notin \varphi(K)$ . Thus we have proved  $\{\mathcal{E}^1(\varphi(L)) \setminus \{\mathbb{R}/I\}\} \subseteq \mathcal{X}(\mathcal{R})$ , and therefore  $\varphi(L) \in \mathcal{R}$  (by 2.9).

2.11. Corollary. The lattice  $\mathcal{L}(\mathbb{R})$  is distributive, and it is complementary iff  $\mathbb{R}$  is a semiartinian ring.

Proof. For  $\mathcal{U}, \mathcal{V} \in \mathcal{L}(\mathbb{R})$  put  $\mathcal{U} \varphi \mathcal{V}$  iff  $\mathcal{F}(\mathcal{U}) = \mathcal{F}(\mathcal{V})$ . From 2.9 it is easy to see that  $\varphi$  is a congruence relation on the lattice  $\mathcal{L}(\mathbb{R})$  and that

$\mathcal{L}(\mathbb{R}) / \varphi \cong \mathcal{L}(\mathbb{R})$ . If, further,  $\mathcal{L}(\mathbb{R})$  is complementary,

then the radical filter  $\mathcal{R}$  which is generated by all maximal left ideals possesses a complement  $\mathcal{T}$ , and consequently  $\mathcal{R} = \mathcal{F}(\mathbb{R})$  (since  $\mathcal{T} \cap \mathcal{R} = \{I\}$  implies  $\mathcal{T} = \{I\}$ ). For the converse implication suppose that  $\mathbb{R}$  is semiartinian and  $\mathcal{U} \in \mathcal{L}(\mathbb{R})$  is an element. Denote  $\mathcal{V} = \{I \mid I \in \mathcal{F}(\mathbb{R}) \text{ is maximal and } I \in \mathcal{U}\}$  and

$\mathcal{X} = \{I \mid I \in \mathcal{S}(R) \text{ is maximal and } I \neq \mathcal{U} \text{ or } I = R\}$ .  
 Obviously,  $\mathcal{X}, \mathcal{V} \in \mathcal{K}(R)$ . Further, since  $R$  is semiartinian,  $\mathcal{F}(\mathcal{V}) = \mathcal{U}$  and  $\mathcal{F}(\mathcal{F}(\mathcal{U}) \cup \mathcal{F}(\mathcal{X})) = \mathcal{S}(R)$ .  
 Finally, let  $\mathcal{F}(\mathcal{V}) \cap \mathcal{F}(\mathcal{X}) \neq \{R\}$ . Then there is  $I \in \mathcal{F}(\mathcal{V}) \cap \mathcal{F}(\mathcal{X})$ ,  $I \neq R$  is a maximal left ideal. By 2.9,  $(I : \lambda) \in \mathcal{X}$  for some  $\lambda \in R \setminus I$ . However,  $1 = \rho\lambda + \alpha$ , where  $\rho \in R$  and  $\alpha \in I$  are suitable, and so  $I = (I : \rho\lambda) = ((I : \lambda) : \rho) \in \mathcal{X}$ . Thus  $I \in \mathcal{X} \cap \mathcal{V}$ , a contradiction.

Let us note here that the preceding corollary was already proved before in [1] for the case of commutative noetherian rings.

3. In this paragraph we generalize some results from [2] to get a characterization of  $\mathcal{F}(\mathcal{M})$ , where  $\mathcal{M}$  is a countable set of two-sided ideals. Let  $\mathcal{M} = \{I_1, I_2, \dots\}$  be a countable subsystem of  $\mathcal{S}(R)$ . A sequence  $\{\lambda_1, \lambda_2, \dots\}$  of elements from  $R$  will be called  $\mathcal{M}$ -regular if the set  $\{i \mid \lambda_i \in I_i\}$  is infinite for any  $i = 1, 2, \dots$ . Denote by  $\mathcal{U}(\mathcal{M})$  the set of all the  $\mathcal{M}$ -regular sequences and put  $\mathcal{G}(\mathcal{M}) = \{I \mid \forall \{\lambda_1, \lambda_2, \dots\} \in \mathcal{U}(\mathcal{M}) \forall \rho \in R \exists n \geq 1 \text{ such that } \lambda_n \dots \lambda_1 \rho \in I\}$ .

3.1. Theorem. Let  $\mathcal{M} = \{I_1, I_2, \dots\}$  be a countable subsystem of  $\mathcal{S}(R)$ . Then:

- (i)  $\mathcal{G}(\mathcal{M})$  is a radical filter.
- (ii)  $\mathcal{G}(\mathcal{M}) \subseteq \mathcal{F}(\mathcal{M})$ .
- (iii)  $\mathcal{G}(\mathcal{M}) = \mathcal{F}(\mathcal{M})$  provided every ideal from  $\mathcal{M}$  is two-sided.

Proof. (i) The condition  $(F_1)$  is obvious. Now  $(F_2)$ .

Let  $I \in \mathcal{G}(\mathcal{M})$  and  $\sigma \in \mathcal{R}$  be arbitrary. If  $\{\lambda_1, \lambda_2, \dots\} \in \mathcal{U}(\mathcal{M})$  and  $\rho \in \mathcal{R}$ , then (by the hypothesis) there is  $n \geq 1$  such that  $\lambda_n \dots \lambda_1 \rho \sigma \in I$ , i.e.  $\lambda_n \dots \lambda_1 \rho \in (I : \sigma)$ . Finally  $(F_6)$ . Let  $I \in \mathcal{F}(\mathcal{R})$ ,  $X \in \mathcal{G}(\mathcal{M})$  and  $(I : \alpha) \in \mathcal{G}(\mathcal{M})$  for each  $\alpha \in X$ . Given  $\{\lambda_1, \lambda_2, \dots\} \in \mathcal{U}(\mathcal{M})$  and  $\rho \in \mathcal{R}$ , there is  $n \geq 1$  with  $\lambda_n \dots \lambda_1 \rho \in X$ . However, the sequence  $\{\lambda_{n+1}, \lambda_{n+2}, \dots\}$  is also  $\mathcal{M}$ -regular and  $(I : \lambda_n \dots \lambda_1 \rho) \in \mathcal{G}(\mathcal{M})$ . Hence there is  $m \geq 1$  such that  $\lambda_{n+m} \dots \lambda_{n+1} \cdot 1 \in (I : \lambda_n \dots \lambda_1 \rho)$  and so  $\lambda_{n+m} \dots \lambda_n \dots \lambda_1 \rho \in I$ .

(ii) Suppose, on the contrary, that there exists  $I \in \mathcal{G}(\mathcal{M})$ ,  $I \notin \mathcal{F}(\mathcal{M})$ . Hence (by  $(F_6)$ ) there is  $\lambda_1 \in I_1$  such that  $(I : \lambda_1) \notin \mathcal{F}(\mathcal{M})$ . Further,  $I_1 \cap I_2 \in \mathcal{F}(\mathcal{M})$  and therefore there is  $\lambda_2 \in I_1 \cap I_2$  such that  $(I : \lambda_2 \lambda_1) = ((I : \lambda_1) : \lambda_2) \notin \mathcal{F}(\mathcal{M})$ . Repeating this argument, we get a sequence  $\{\lambda_1, \lambda_2, \dots\}$  having the following properties:

( $\alpha$ )  $\lambda_j \in I_1 \cap I_2 \cap \dots \cap I_j$  for every  $j = 1, 2, \dots$ ,

( $\beta$ )  $(I : \lambda_j \dots \lambda_1) \notin \mathcal{F}(\mathcal{M})$  for every  $j = 1, 2, \dots$ .

From ( $\alpha$ ) we see that  $\{\lambda_1, \lambda_2, \dots\}$  is an  $\mathcal{M}$ -regular sequence. Hence, by the hypothesis,  $\lambda_n \dots \lambda_1 \cdot 1 \in I$  for some  $n \geq 1$ , and consequently  $(I : \lambda_n \dots \lambda_1) = \mathcal{R}$ , which is a contradiction with ( $\beta$ ).

(iii) Obvious, since  $I_j \in \mathcal{G}(\mathcal{M})$  whenever  $I_j$  is a two-sided ideal.

3.2. Corollary. Let  $\mathcal{M} = \{I_1, \dots, I_m\}$  be a finite set of two-sided ideals. Then  $\mathcal{F}(\mathcal{M}) = \{I \mid \forall \lambda_1, \lambda_2, \dots \in I_1 \cap \dots \cap I_m \exists m \geq 1 \text{ such that } \lambda_m \dots \lambda_1 \in I\}$ .

Proof. Denote by  $\mathcal{J}$  the set defined above. From 3.1 it is obvious that  $\mathcal{F}(\mathcal{M}) \subseteq \mathcal{J}$ . In order to prove the converse inclusion we need only to observe the following fact. If  $\{\lambda_1, \lambda_2, \dots\} \in \mathcal{U}(\mathcal{M})$ , then there exist  $1 \leq l_1 < l_2 < l_3 < \dots$  such that  $\lambda_{l_j} \cdot \lambda_{l_j-1} \dots \lambda_{l_j-1} \in I_1 \cap \dots \cap I_m$  for all  $j = 1, 2, \dots$ .

3.3. Corollary. Let  $\mathcal{M}$  be a finite set of two-sided ideals. Then  $0 \in \mathcal{F}(\mathcal{M})$  iff  $\bigcap_{I \in \mathcal{M}} I$  is right T-nilpotent.

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