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CONCERNING THE STRUCTURE OF DENDRITIC SPACES

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Abstract: A dendritic space is a nondegenerate connected Hausdorff space such that each two of its points are separated by a third point. In this paper we obtain some structure theorems for general dendritic spaces and for dendritic spaces satisfying certain weak compactness conditions stated in terms of the convergence of nets of point sets.

Key words and phrases: Dendritic spaces, tree-like spaces, arcs in dendritic spaces, nets of point sets.

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1. Definition. Suppose $\{U_n, n \in D\}$ is a net of point sets in a topological space. Then lim sup U_n is the set of all points x such that for each open set U containing x and each m there is an $n \geq m$ such that $U_n \cap U \neq \emptyset$, and lim inf U_n is the set of all points x such that for each open set U containing x there is an m such that if $n \geq m$ then $U_n \cap U \neq \emptyset$.

It should be noted that it does not follow, even for sequences, that if $x \in$ lim inf U_n , then for each n there exists a point x_n of U_n such that x is a cluster point of the net $\{x_n, n \in D\}$. Consider the following counterexample.

Example. For each positive integer n let U_n be the set of all ordered pairs (n, m) of integers such that $0 \leq m \leq n$, let $x = (0, 0)$ and let $X = \bigcup_n U_n \cup \{x\}$. Let $\mathcal{S}$ be the collection of all point sets U in X such that either $x \notin U$ or $x \in U$ and $X - U$ is a choice set for the collection of all U_n, i.e., for each n there exists a point x_n of U_n such that $U_n - U = \{x_n\}$. Take $\mathcal{S}$ as a subbase for the open sets in X. Thus X is a Hausdorff space, and $X - \{x\}$ is discrete. Furthermore, $\{x\} = \liminf U_n = \limsup U_n$, but x is not a cluster point of any sequence $x_1, x_2, \ldots$ such that for each n, $x_n \in U_n$.

The proofs of the following fundamental theorems parallel the proofs of similar theorems on nets of points and are omitted.

Theorem 1. If $\{U_n, n \in D\}$ is a net of point sets with the point x in its $\limsup$, then some subnet of $\{U_n, n \in D\}$ has x in its $\liminf$.

Theorem 2. If $\mathcal{S}$ is a collection of point sets and x is a limit point of $\bigcup\mathcal{S}$, then there exists a net of elements of $\mathcal{S}$ having x in its $\liminf$.

Some very general classes of topological spaces may be defined by stipulating that certain nets of point sets of a certain sort have a non-empty $\limsup$. In what follows we consider one such class of spaces, which is of interest in connection with dendritic spaces. If M is a point set, then the boundary of M is denoted by ∂M and the cardinal of M is denoted by $|M|$.

Definition. If X is a topological space and k is a cardinal, then X is k-cohesive if and only if the following condition holds. If $\{U_n, n \in D\}$ is a net of connected open sets in X such that (1) if $m \neq n$, then $U_m \cap U_n = \emptyset$ or $U_m = U_n$, (2) for each n, $0 < |\partial U_n| \leq k$, and (3) $\liminf U_n \neq \emptyset$, then $\limsup \partial U_n \neq \emptyset$.

Theorem 3. If the space X is either compact or locally connected, then for each finite cardinal k, X is k-cohesive.

Proof. Let $\{U_n, n \in D\}$ be a net satisfying the conditions of the above definition, and for each n let $\partial U_n = \{x_{n1}, \ldots, x_{nk}\}$. If there is an m such that for each $n \geq m$, $U_n = U_m$, then clearly $\limsup \partial U_n \neq \emptyset$. Hence we assume that for each m there is an $n > m$ such that $U_n \neq U_m$. Let $x \in \liminf U_n$. Thus for each n, $x \in X - U_n$. If X is compact, then the net $\{x_{n1}, n \in D\}$ has a cluster point y and hence $y \in \limsup \partial U_n$. Suppose X is locally connected and x is not a cluster point of $\{x_{ni}, n \in D\}$ for $i = 1, \ldots, k$. For each i there exist an open set V_i containing x and an m_i such that if $n \geq m_i$, then $x_{ni} \notin V_i$. Let V be a connected open set containing x and lying in $\bigcap_{i=1}^{k} V_i$, and let $n \in D$ such that for each i, $n \geq m_i$ and $U_n \cap V \neq \emptyset$. Since V is connected and contains both a point of U_n and a point of $X - U_n$, V contains a point of ∂U_n, which is a contradiction. Therefore x is a cluster point of $\{x_{ni}, n \in D\}$ for some i, and hence $\limsup \partial U_n \neq \emptyset$.

2. Dendritic spaces

Theorem 4. If x is a point of the dendritic space X and U is a component of $X - \{x\}$, then U is open and x is a limit point of U.

Proof. Let $y \in U$. There is a point p such that $X - \{p\}$ is the union of two disjoint open sets V and W such that $x \in V$ and $y \in W$. Since $W \cup \{p\}$ is connected and does not contain x, $W \cup \{p\} \subseteq U$. Hence U is open. Since X is connected and each component of $X - \{x\}$ is open, x is a limit point of U.

Theorem 5. For each two points x and y of the dendritic space X there exists one and only one component of $X - \{x, y\}$ with x and y as limit points.

Proof. Let C be the component of $X - \{x\}$ containing y. From Theorem 4, x is a limit point of C. There is a point p of X such that $X - \{p\}$ is the union of two disjoint open sets U and V with $x \in U$ and $y \in V$. Clearly, $p \in C - \{y\}$. Let K be the component of $C - \{y\}$ containing p. Now C is dendritic, and hence y is a limit point of K. Since $C - K$ is a connected subset of $X - \{p\}$ containing y, $C - K \subseteq V$. Hence x is not a limit point of $C - K$, so that x is a limit point of K. Suppose H is a connected subset of $X - \{x, y\}$ containing K. Since $p \in H \subseteq X - \{x\}$, $H \subseteq C$. Since $p \in H \subseteq C - \{y\}$, $H \subseteq K$. Hence $H = K$, so that K is a component of $X - \{x, y\}$ with limit points x and y. If L is a component of $X - \{x, y\}$ different from K with limit points x and y, then $L \cap K = \emptyset$, and hence no point of X separates x from y.

Definition. If a and b are two points of the connected space X, then the interval ab of X, denoted simply by ab, is the set of all points x of X such that $x = a$, $x = b$ or x separates a from b in X. If $x, y \in ab$, then $x < y$ if and only if $x = a$ and $y \neq a$, or x separates a from $\{y, b\}$ in X.

Simple examples may be given to show that intervals in dendritic spaces may be neither compact nor connected. It follows from the next theorem that they are, however, closed. An example is then given of a dendritic space in which each interval is totally disconnected.

Theorem 6. If a and b are points of the dendritic space X, then there exists a collection $\mathcal{S}$ of disjoint connected open sets in X such that $X - \bigcup\mathcal{S} = ab$ and for each element U of $\mathcal{S}$ there exists a point x of ab such that $\partial U = \{x\}$.

Proof. For each point x of ab let $\mathcal{S}_x$ denote the set of all components C of $X - \{x\}$ such that C contains neither a nor b and let $U_x = \bigcup\mathcal{S}_x$. Let $\mathcal{S} = \bigcup\{\mathcal{S}_x \mid x \in ab\}$. It follows from Theorem 5 that for each two points x and y of X there is a unique component C_{xy} of $X - \{x, y\}$ that has x and y as limit points and such that if x and y are in ab, then $x < y$. Let $K_{xy} = C_{xy} \cup \{x, y\}$. Suppose $p \in X - (\bigcup\mathcal{S} \cup ab)$. Since $X = U_a \cup U_b \cup K_{ab}$, $p \in K_{ab}$. Let $\mathcal{J}$ be the collection of all K_{xy} such that $a \leq x < y \leq b$ and $p \in K_{xy}$. Partially order $\mathcal{J}$ by set inclusion. Let $\mathcal{P}$

be a maximal chain in $\mathcal{C}$, and let $K = \bigcap \mathcal{P}$. If $a \leq x \leq b$, let V_x be the component of $X - \{x\}$ containing a if $x \neq a$ and let $V_x = \emptyset$ if $x = a$, and let W_x be the component of $X - \{x\}$ containing b if $x \neq b$ and let $W_x = \emptyset$ if $x = b$.

Case 1. For each two points w and z such that $a \leq w < z \leq b$ and $K_{wz} \in \mathcal{P}$ there exist two points x and y such that $w < x < y < z$ and $K_{xy} \in \mathcal{P}$. Let V be the union of all V_x such that for some point y, $a < x < y < b$ and $K_{xy} \in \mathcal{P}$. Let W be the union of all W_y such that for some point x, $a < x < y < b$ and $K_{xy} \in \mathcal{P}$. $X - K_{wz} = (V_w \cup U_w) \cup (W_z \cup U_z) \subseteq V_x \cup W_y$. Furthermore, V_x and W_y are disjoint connected open sets containing a and b respectively. It follows that $X - K = V \cup W$ and V and W are disjoint connected open sets containing a and b respectively. Suppose K contains two boundary points x and y of V. Some point q of X separates x from y. Since $V \cup \{x, y\}$ is connected, $q \in V$. For some w and z such that $K_{wz} \in \mathcal{P}$, $q \in V_w$. But K_{wz} is a connected subset of $X - \{q\}$ containing x and y. Therefore K contains only one boundary point x of V and only one boundary point z of W. Clearly, $a < x < z < b$ and $K = K_{xz}$. There exists a point y such that $x < y < z$. Now $p \in K_{xy}$ or $p \in K_{yz}$, say $p \in K_{xy}$. Hence $K_{xy} \in \mathcal{C}$, and K_{xy} is a proper subset of every element of $\mathcal{P}$, which contradicts the maximality of $\mathcal{P}$.

<u>Case 2</u>. There exists a point w such that $a \leq w < b$ and for each two points x and y such that $w \leq x < y \leq b$ and $K_{xy} \in \mathcal{P}$, $x = w$. Let $V = V_w \cup U_w$. Let W be the union of all W_y such that $w < y < b$ and $K_{wy} \in \mathcal{P}$. It follows that $X - K = V \cup W$, V and W are disjoint open sets, W is connected, $b \in W$, and if $w \neq a$, $a \in V$. We again arrive at a contradiction if K contains two boundary points of W. Hence K contains only one boundary point z of W. Clearly, $a \leq w < z < b$ and $K = K_{wz}$. There exists a point y such that $w < y < z$. $p \in K_{wy}$. K_{wy} is then a proper subset of every element of $\mathcal{P}$, which is a contradiction.

<u>Case 3</u>. There exists a point z such that $a < z \leq b$ and for each two points x and y such that $a \leq x < y \leq z$ and $K_{xy} \in \mathcal{P}$, $y = z$. Thus Case 3 is similar to Case 2.

<u>Example.</u> Let X be the set of all points $x = x_1, x_2, \dots$ of Hilbert space such that $x_1 > 0$, for each n, $x_n \geq 0$, and for all but finitely many n, $x_n = 0$. For each point x of X and for each positive number r let i be the largest integer m such that $x_m > 0$ and let D_{xr} be the set of all points y of X such that (1) $x_m = y_m$ for $m \neq i$ and $m \neq i+1$, (2) $0 \leq |x - y| < r$, and (3) if $x \neq y$, $y_{i+1} > 0$. Thus D_{xr} is the intersection with X of a semicircular region together with the point x. Note that if $y \in D_{xr}$, $y \neq x$, and $s > 0$, then $D_{xr} \cap D_{ys} = \{y\}$ and $D_{xr} \perp D_{ys}$. For each x in X and each map f of X into the positive reals let $U_0 = \{x\}$, let $U_1 = D_{x,f(x)}$, for each $n > 1$ let

$\mathcal{U}_m = \bigcup\{D_{y,f(y)} \mid y \in \mathcal{U}_{m-1} - \mathcal{U}_{m-2}\}$, and let $\mathcal{U}_{xf} = \bigcup_n \mathcal{U}_n$. Now if $z \in \mathcal{U}_{xf} \cap \mathcal{U}_{yg}$ and $h = f \wedge g$, then $\mathcal{U}_{zh} \subseteq \mathcal{U}_{xf} \cap$ $\cap\, \mathcal{U}_{yg}$. The collection of all such sets $\mathcal{U}_{xf}$ is then taken as a base for the open sets in X . In order to show that X is connected suppose X is the union of two disjoint open sets U and V and first show that the set of all points x of X such that $x_m = 0$ for $m > 1$ is a subset of one of the two sets U and V , say U , then show that for each positive number t the set of all points x of X such that $x_1 = t$ and $x_m = 0$ for $m > 2$ is a subset of U , and finally conclude that $V = \emptyset$. Now suppose a and b are two points of X , $a_m = 0$ for $m > 1$, and k is the largest integer m such that $b_m \neq \emptyset$. For $m = 0, \dots, k$ let $p^m \in X$ such that $p^0 = a$ and for $m > 0$, $p^m_i = b_i$ for $i \leq m$ and $p^m_i = 0$ for $i > m$. The interval ab of X is the union of the straight line intervals $[p^m, p^{m+1}]$ of Hilbert space for $m = 0, \dots$ $\dots, k-1$. In the space X , p is a limit point of ab if and only if for some m such that $0 < m < k$, $p = p^m$. Similar considerations will show that each interval of X has at most finitely many limit points and hence is totally disconnected.

Theorem 7. If X is dendritic and 1-cohesive, then each interval of X is connected.

Proof. Let a and b be two points of X , and suppose ab is not connected. Since it follows from Theorem 6 that ab is closed, ab is the union of two disjoint closed sets H and K . Let $\mathcal{Y}$ be the collection mentioned in Theorem 6, let $\mathcal{G}_H = \{U \in \mathcal{G} \mid \partial U \subseteq H\}$, let $\mathcal{G}_K = \{U \in \mathcal{G} \mid \partial U \subseteq K\}$, let $H' = \bigcup \mathcal{G}_H$, and let $K' = \bigcup \mathcal{G}_K$. Since X is connected, some point of H is a limit point of K' or some point of K is a limit point of H'. Assume the point p of H is a limit point of K'. It follows from Theorem 2 that there exists a net $\{U_n, n \in D\}$ of elements of $\mathcal{G}_K$ having p in its *lim inf*. Now since for each n, $|\partial U_n| = 1$ and X is 1-cohesive, there exists a point q in *lim sup* $\{\partial U_n, n \in D\}$. Since K is closed, $q \in K$, so that $p \neq q$. From Theorem 1, some subnet $\{\partial U_{n_i}, i \in E\}$ of $\{\partial U_n, n \in D\}$ has q in its *lim inf*. Thus $\{U_{n_i}, i \in E\}$ is a net whose range is a collection of disjoint connected sets in X such that both p and q are in its *lim inf* and for each i, $p \in X - \overline{U_{n_i}}$. It follows that no point separates p from q in X, which is a contradiction.

<u>Theorem 8</u>. If a and b are two non-cut points of the connected 2-cohesive space X and every point of $X - \{a, b\}$ separates X into two connected sets one containing a and the other containing b, then X is an arc from a to b.

<u>Proof</u>. It follows from known results without the use of 2-cohesiveness that X is an arc from a to b in its order topology and that each order interval of X of the form $[a, x)$, (x, y), or $(y, b]$ is open and connected in the original topology of X. It remains to be shown that the original topology of X has a base whose elements are intervals of the above type. Let U be an open set contai-

ning b , and suppose that for each x in (a, b), $(x, b] - U \neq \emptyset$. There exists an increasing well-ordered sequence $\{x_\alpha, \alpha < \lambda\}$ of points of $X - U$ converging to b in the order topology. For each α let $U_\alpha = (x_\alpha, x_{\alpha+1})$. Since $(x_0, b]$ is open, b is not a limit point of $[a, x_0)$, and hence b is a limit point of $\bigcup_\alpha U_\alpha$. Furthermore, for each α, $\partial U_\alpha = \{x_\alpha, x_{\alpha+1}\}$. Therefore there exists a net $\{U_n, n \in D\}$ of elements of $\{U_\alpha \mid \alpha < \lambda\}$ having b in its $\liminf$. But $b \notin \limsup\{\partial U_n, n \in D\}$, which contradicts the assumption that X is 2-cohesive. Therefore $(x, b]$ is open, and similar considerations will show that intervals of the type $[a, x)$ and (x, y) are open.

<u>Theorem 9</u>. If X is dendritic and 2-cohesive, then each interval of X is an arc.

<u>Proof</u>. Let a and b be two points of X . From Theorem 7, ab is connected. If $a < y < b$, $[a, y) = \bigcup\{[a, x] \mid a < x < b\}$ and hence $[a, y)$ is connected. Similarly, $(y, b]$ is connected. Therefore $ab - \{y\}$ is the union of two disjoint connected sets one containing a and the other containing b . Let $\mathcal{G}$ be the collection mentioned in Theorem 6, and for each subset U of ab let $U' = U \cup \bigcup\{V \in \mathcal{G} \mid \partial V \subseteq U\}$. It is easily seen that if U is open and connected relative to ab , then U' is open and connected and that for each point p , p is a boundary point of U relative to ab if and only if p is a boundary point of U' . Furthermore, if $U, V \subseteq ab$, then $U \cap V = \emptyset$ if and only if $U' \cap V' = \emptyset$. Therefore the

2-cohesiveness of X implies the 2-cohesiveness of ab. It then follows from Theorem 7 that ab is an arc.

Example. Let X be the set of all points (x,y) in the plane such that $0 < x \leq 2\pi$ and $y = \sin 1/x$ together with the point $(0,0)$. In its subspace topology X is dendritic and 1-cohesive but is not 2-cohesive, and X is an interval of itself but is not an arc.

Theorem 10. If X is dendritic, arcwise connected, and 1-cohesive, then X is locally connected.

Proof. Suppose X is not connected im kleinen at the point p. Then there is an open set U containing p such that for each open set V containing p and lying in U there is a point z of V such that no connected set containing both p and z lies in U. Hence there exists an indexed set $M = \{z_\alpha \mid \alpha \in A\}$ of points of $U - \{p\}$ such that p is a limit point of M and for each α in A the arc pz_α intersects $X - U$. For each α let x_α be the first point of ∂U on pz_α, let $y_\alpha \in X - U$ such that $x_\alpha y_\alpha z_\alpha$ and let C_α be the component of $X - \{x_\alpha\}$ containing z_α. Now for each α, x_α separates p from z_α, so that $C_\alpha \cap px_\alpha = \emptyset$. Hence if $x_\alpha \neq x_\beta$ and $C_\alpha \cap C_\beta \neq \emptyset$, then $px_\alpha \cup px_\beta$ and $C_\alpha \cup C_\beta$ are two connected sets with intersection $\{x_\alpha, x_\beta\}$ and therefore no point of X separates x_α from x_β. It follows that for each α and β in A either $C_\alpha = C_\beta$ or $C_\alpha \cap C_\beta = \emptyset$. Since X is 1-cohesive, there is a net $\{\partial C_n, n \in D\}$ of elements of $\{\partial C_\alpha \mid \alpha \in A\}$ with a point q in its $\lim\sup$. Hence some subnet $\{C_{n_i}, i \in E\}$ of $\{C_n, n \in D\}$ has both p and q in its $\liminf$, and $p \neq q$ since $p \in U$ and $q \in X - U$. It follows that no point of X separates p from q. Therefore X is connected im kleinen at each of its points and hence is locally connected.

Theorem 11. In order that the dendritic space X be locally connected, it is necessary and sufficient that it be 2-cohesive.

Proof. Theorem 11 follows from Theorems 3, 9, and 10.

Theorem 12. If the dendritic space X is 2-cohesive, then for each finite n, X is n-cohesive.

Proof. Theorem 12 follows from Theorems 3 and 11.

It follows from Theorems 9 and 11 that every locally connected dendritic space is arcwise connected, a result which is already known from Whyburn's extension of his cyclic element theory to non-metric spaces in [3] and which is mentioned by Proizvolov [2] and attributed to Gurin [1]. In connection with Theorem 11 we note that Gurin [1] proved that in order that a dendritic space be locally connected it is sufficient that if be locally peripherally compact. The condition is not, however, necessary.

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