Svatopluk Poljak
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A NOTE ON STABLE SETS AND COLORINGS OF GRAPHS

Svatopluk POLJAK, Praha

Abstract: It is given here an explicit reduction of the problem of determining the stability number $\alpha(G)$ of a graph G into the problem of determining the chromatic number $\chi(H)$ of a graph H . This is related to the Karp-Cook complexity theory.

Key words: Graph, chromatic number, stability number, computational complexity.

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The problems of computing the stability number $\alpha(G)$ and the chromatic number $\chi(G)$ of a graph G belong to the most difficult combinatorial problems (for the graph-theoretical definitions, see [1]). It seems very unlikely that there is an efficient algorithm (that is, an algorithm terminating within $p(n)$ steps where $p(\cdot)$ is a fixed polynomial and n is the number of vertices of G) for computing either of these invariants. This sentiment is supported by results of Cook [3] and Karp [4]. Indeed, it follows immediately from the Cook's theorem that the existence of an efficient algorithm for computing $\alpha(G)$ would imply the existence of efficient algorithms for all the problems in an extremely wide class (called NP and including every problem solvable by polynomial-depth backtrack search). Besides, it follows from Karp's considerations that there is an efficient algorithm for computing $\alpha(G)$ if and only if there is an efficient algorithm for computing $\chi(G)$. Since Karp's proof is indirect (he reduces a certain logical problem into each of our problems and then applies the Cook's theorem), it may be of interest to exhibit explicit reductions of one into another.

The reduction of $\chi(G)$ into $\alpha(H)$ has been mentioned by Chvátal [2]: if G has n vertices then $\chi(G) \leq x$ if and only if $\alpha(G \times K_x) = n$. We shall describe a reduction of $\alpha(G)$ into $\chi(H)$. Let G be a graph with n vertices and m edges. Replace each edge of G by a path consisting of three edges; call the resulting graph F . Then F has $n + 2m$ vertices and $\alpha(F) = \alpha(G) + m$. Next, construct a graph H whose vertices correspond to the edges of F ; two vertices of H are adjacent if and only if the corresponding edges of F do not share an endpoint. Since F contains no triangle, we have

$$\chi(H) = (2m + n) - \alpha(F) = m + n - \alpha(G) \ .$$

Clearly, the construction of H can be carried out within $O(n^4)$ steps.

Many thanks to V. Chvátal!

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Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83, 18600 Praha 8

Československo

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