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A NOTE ON MARKUŠEVIČ BASES IN WEAKLY COMPACTLY GENERATED BANACH SPACES

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Abstract: The concept of Markuševič bases is used to give more elementary proofs of some results on weakly compactly generated Banach spaces.

Key words: Weakly compactly generated Banach spaces, Markuševič basis.

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Notation: By normed linear space, we shall mean a real one, by $sp\, A$ for a set A we denote the linear span of A, $\overline{sp}\, A$ denotes the closure of $sp\, A$.

For a locally convex space X, by X^* we mean the dual of X (i.e. continuous linear functions on X).

A Banach space X is called weakly compactly generated (in short WCG) if there exists a weakly compact set $K \subset X$ such that $\overline{sp}\, K = X$.

A biorthogonal system $\{x_i, f_i\}_{i \in I}$ in $X \times X^*$ (X is in general a locally convex space) is called a Markuševič basis (in short M-basis) if $f_i(x_j) = \delta_{ij}$ for $i, j \in I$, $\overline{sp}\{x_i\}_{i \in I} = X$, and $f_i(x) = 0$ for all $i \in I$ implies $x = 0$. By w (resp. w^*) we mean the $\sigma(X, X^*)$ (resp. $\sigma(X^*, X)$) topology. $c_0(\Gamma)$ denotes the space of real valued functions x on Γ such that for each $\varepsilon > 0$ the set $\{\gamma \in \Gamma ; |x(\gamma)| > \varepsilon\}$ is finite, $\|x\| = \sup_{\gamma} |x(\gamma)|$.

ω is the first infinite ordinal number. For a topological space X we denote by $w X$ the smallest cardinal number $\aleph$ such that there exists a set $A \subset X$, $\operatorname{card} A = \aleph$, A is dense in X. We say that a locally convex space is generated by A if $X = \overline{sp}\, A$.

<u>Lemma</u> ([1]). Let X be a WCG space, generated by a weakly compact absolutely convex set K. Denote ς the first ordinal of cardinality wX. Then there exists a system $\{P_\alpha\}_{\omega \leq \alpha \leq \varsigma}$ of linear projections such that $\|P_\alpha\| = 1$, $P_\alpha K \subset K$, $w(P_\alpha X) = \operatorname{card} \alpha$ for each α, $P_\varsigma = id_X$, $P_\alpha P_\beta = P_\beta P_\alpha = P_\alpha$ for each $\alpha \leq \beta$, $P_\alpha X = \overline{\bigcup_{\beta < \alpha} P_\beta X}$ for each α limit ordinal.

<u>Proposition</u>. Let X be a WCG space and $K \subset X$ be a weakly compact absolutely convex set generating X. Then there exists a Markuševič basis $\{x_i, f_i\}_{i \in I}$ of X such that $x_i \in K$ for $i \in I$.

Proof. We prove Proposition by transfinite induction on wX. Let X be separable. Then $sp\, K$ is a separable normed linear space and there exists an M-basis of $sp\, K$, which is also an M-basis of X (see e.g. [3]). For this M-basis $\{x_i, f_i\}_{i \in I}$ we can suppose $x_i \in K$ as K absorbs elements of $sp\, K$. Suppose now that $wX > \aleph_0$ and that Proposition has been proved for all spaces Y with $wY < wX$. Let $\{P_\alpha\}_{\omega \leq \alpha \leq \varsigma}$ be the system of pro-

jections from the above lemma. Then $Y_\omega = P_\omega X$ is a WCG space generated by weakly compact absolutely convex set $K_\omega = P_\omega K$, $wY_\omega = \operatorname{card} \omega < wX$. Similarly $Y_{\alpha+1} = (P_{\alpha+1} - P_\alpha)X$ is generated by weakly compact absolutely convex set $K_{\alpha+1} = \frac{1}{2}(P_{\alpha+1} - P_\alpha)K$ for $\alpha < \xi$, and $wY_\alpha < wX$ for all these α. By the induction hypothesis there exists for each $\alpha < \xi$ an M-basis $\{x_i^\alpha, f_i^\alpha\}_{i \in I_\alpha}$ of Y_α with $x_i^\alpha \in K_\alpha$ $(i \in I_\alpha)$.
Now $\{x_i^\omega, f_i^\omega \circ P_\omega\}_{i \in I_\omega} \cup \{x_i^\alpha, f_i^\alpha \circ (P_{\alpha+1} - P_\alpha)\}_{i \in I_\alpha, \omega < \alpha < \xi}$ is obviously an M-basis of X and $\{x_i^\alpha\}_{i \in I_\alpha, \alpha < \xi} \subset \bigcup_{\alpha < \xi} K_\alpha \subset K$.

Corollary 1. A Banach space is WCG if and only if there exists an M-basis $\{x_i, f_i\}_{i \in I}$ of X such that $\{x_i\}_{i \in I} \cup \{0\}$ is weakly compact. The coefficients $\{x_i\}_{i \in I}$ of such an M-basis in a WCG space X can be found in an arbitrary weakly compact absolutely convex set K generating X.

Proof. The part "if" of our assertion is trivial as X is generated by $\{x_i\}_{i \in I}$.
Let X be a WCG space and K be a set as above. By Proposition there exists an M-basis $\{x_i, f_i\}_{i \in I}$ of X such that $A = \{x_i\}_{i \in I} \subset K$. We must only prove that the only w-cluster point of A is $x = 0$ (A is obviously discrete in w topology). Let x be a cluster point of A, and x be the limit of a net $\{x_\nu\}_{\nu \in \Lambda} \subset A$. Then for an arbitrary $i \in I$ is $x_\nu \neq x_i$ for $\nu \geq \nu_0$ for some $\nu_0 \in \Lambda$. Thus $f_i(x) = \lim_\nu f_i(x_\nu) = 0$ which implies

$x = 0$ as $\{f_i\}_{i \in I}$ is total on X .

Remark 1. The following two results are due to Amir, Corson and Lindenstrauss ([1],[4]). However, they used for their proofs a measure representation theorem and the Stone-Weierstrass theorem.

Corollary 2. Every WCG space is generated by a set which is in the weak topology one point compactification of a discrete set. This set can be found in an arbitrary weakly compact absolutely convex set K generating the space.

Proof. For the generating set take the set $\{x_i\}_{i \in I} \cup \cup \{0\}$ from Corollary 1.

Corollary 3. Let X and $\{x_i, f_i\}_{i \in I}$ be as in Corollary 1. Then the mapping $T: X^* \longrightarrow c_0(I)$ defined by $T(f) = f\{x_i\}_{i \in I}$ is a 1-1 linear w^*-w continuous mapping onto a dense subset of $c_0(I)$.

Proof. The w^*-w continuity follows from the theorem of Banach-Dieudonné.

Remark 2. Let X be a locally convex space and $\{x_i, f_i\}_{i \in I}$ an M -basis of X such that $K = \{x_i\}_{i \in I}$ is relatively w compact. Then the mapping $T: X^* \longrightarrow c_0(I)$, $T(f) = \{f(x_i)\}_{i \in I}$ is a 1-1 linear mapping which is w^*-w continuous on $K^0 = \{f \in X^*; |f(k)| \leqq 1$ for $k \in K\}$.

Two following simple examples satisfy assumptions of

Remark 2.

Example 1. Let $\mathcal{Y}$ be a normed linear space with an $\mathcal{M}$ -basis $\{x_i, f_i\}_{i\in I}$. We can suppose that $\|f_i\| \leqq 1$ for $i \in I$. Denote $\mathcal{X} = \mathcal{Y}^*$ with some topology which coincides with the duality $\langle \mathcal{Y}, \mathcal{Y}^* \rangle$. Then $\{f_i, x_i\}_{i\in I}$ is an $\mathcal{M}$ -basis of $\mathcal{X}$.

Example 2. Let $\mathcal{Y}$ be as in Example 1. We can suppose that $\|x_i\| \leqq 1$ for $i \in I$ (f_i can be unbounded). Let $Z \subset \mathcal{Y}^*$ be a subset, $Z \supset \{f_i\}_{i\in I}$, such that $\{x_i\}_{i\in I}$ is $\sigma(\mathcal{Y}, Z)$ relatively compact (for example $Z = \overline{sp}\{f_i\}_{i\in I}$). Denote $\mathcal{X} = \mathcal{Y}$ with some topology which coincides with the duality $\langle \mathcal{Y}, Z \rangle$.

Theorem 7 of [2] due to C. Bessaga, A. Pelczyński and S. Trajanski and Proposition 3,4 of [4] due to H. Corson (see Corollary 2 above) combine to give

Corollary 4. Let $\mathcal{K}$ be an absolutely convex weakly compact subset of a Banach space $\mathcal{X}$. Then the space $C(\mathcal{K})$ of all continuous (with respect to w topology on $\mathcal{K}$) real valued functions on $\mathcal{K}$ is homeomorphic to the space $\ell_1(\xi)$ where $\xi = w(C(\mathcal{K})) = w\mathcal{K}$.

Proof. Denote $\mathcal{Y} = \overline{sp}\,\mathcal{K}$, and ξ be the first ordinal of cardinality $w\mathcal{Y}$. (There is $w\mathcal{Y} \leqq w\mathcal{K}$.) Let L be the generating set of $\mathcal{Y}$ in $\mathcal{K}$ from Corollary 2. L is one point compactification of a discrete set and so $C(L)$ is homeomorphic to $\ell_1(\xi)$. (Trojanski (Th.7,[2])). As $\mathcal{Y}$ is a normed linear space there exists a set $Z \subset \mathcal{Y}^*$,

$\mathrm{card}\ Z = \mathrm{card}\ \varsigma$, Z separates points of Y. Thus we have $L \subset K$, $C(L) \simeq \ell_1(\varsigma)$ and $w(C(K)) \geq \mathrm{card}\ \varsigma$ and thus $w(C(K)) = \mathrm{card}\ \varsigma$ by Stone-Weierstrass theorem. Therefore $C(K)$ is homeomorphic to $\ell_1(\varsigma)$ (Th.7,[21).

R e f e r e n c e s

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