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Generalized symmetric spaces (Preliminary communication)


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GENERALIZED SYMMETRIC SPACES

(Preliminary communication)

Oldřich KOWALSKI, Praha

Abstract: In this note we give some new results concerning generalized symmetric Riemannian spaces (i.e., Riemannian manifolds which admit a regular family of symmetries in the sense of A.J. Ledger). We also present a complete classification of all simply connected irreducible generalized symmetric spaces of dimension 3, 4 and 5 that are not symmetric in the sense of E. Cartan.

Key words: Homogeneous manifolds, Riemannian manifolds, symmetric spaces.

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Let \((M, \phi)\) be a smooth Riemannian manifold. An \(\alpha\)-structure on \((M, \phi)\) is a family \(\{\alpha_x : x \in M\}\) of isometries of \((M, \phi)\) (called symmetries) such that each \(\alpha_x\) has the point \(x\) as an isolated fixed point. The corresponding tensor field \(S\) of type \((1,1)\) defined by \(S_x = (\alpha_x)_x\) for each \(x \in M\) is called the symmetry tensor field of \(\{\alpha_x\}\). Follow-

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With respect to the special character of the paper, Editorial Board agreed with the unusual size of this preliminary communication.

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ing A.J. Ledger ([1],[2]) an $\mathcal{A}$-structure $\{\mathcal{A}_x\}$ on $(M, g)$ is called regular if for every pair of points $x, y \in M$

$$\mathcal{A}_x \circ \mathcal{A}_y = \mathcal{A}_z \circ \mathcal{A}_x$$

where $z = \mathcal{A}_x(y)$.

An equivalent condition is that the corresponding tensor field $S$ is invariant by each $\mathcal{A}_x$, i.e., for all $x \in M$ and all vector fields $X$ on $M$

$$\mathcal{A}_x(SX) = S(\mathcal{A}_x X).$$

From a result by F. Brickel ([3], Theorem 1) we can obtain:

**Theorem 1.** For a regular $\mathcal{A}$-structure $\{\mathcal{A}_x\}$ on $(M, g)$ the symmetry tensor field $S$ is always smooth.

An $\mathcal{A}$-structure $\{\mathcal{A}_x\}$ is called of order $\lambda$ ($\lambda \geq 2$) if, for all $x \in M$, $(\mathcal{A}_x)^{\lambda} = \text{identity}$, and $\lambda$ is the least integer of this property.

Using an unpublished result by A.W. Deicke we can prove

**Theorem 2.** If the Riemannian manifold $(M, g)$ admits a regular $\mathcal{A}$-structure then it also admits a regular $\mathcal{A}$-structure of finite order.

On account of Theorem 2 we introduce

**Definition 1.** A generalized symmetric space (g.s. space) is a Riemannian manifold $(M, g)$ admitting a regular $\mathcal{A}$-structure. Order of a g.s. space $(M, g)$ is the least integer $\lambda$ such that $M$ admits a regular $\mathcal{A}$-structure of order $\lambda$.

Let us remark that the g.s. spaces of order 2 are

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nothing but the Cartan symmetric spaces, and the g.s. spaces of dimension 2 are homogeneous spaces of constant curvature.

Let $(M, \varphi)$ be a g.s. space and $\{\sigma_x\}$ a fixed regular $\varphi$-structure on $(M, \varphi)$. Then the triplet $(M, \varphi, \{\sigma_x\})$ will be called a (Riemannian) $\varphi$-manifold. Let now $\nabla$ denote the Riemannian connection of $(M, \varphi)$ and $\mathcal{S}$ the symmetry tensor field of $\{\sigma_x\}$. Following A.J. Ledger [1], we introduce a new linear connection $\tilde{\nabla}$ by the formula

$$\tilde{\nabla}_X Y = \nabla_X Y - D(Y, X),$$

where

$$D(Y, X) = (VS)(S^{-1}Y, (I-S)^{-1}X) = (\nabla_{(I-S)^{-1}X}S)(S^{-1}X).$$

The basic properties of the connection $\tilde{\nabla}$ are the following:

1) All symmetries $\sigma_x$, $x \in M$, are affine transformations of the affine manifold $(M, \tilde{\nabla})$.

2) The affine manifold $(M, \tilde{\nabla})$ is complete.

3) $(M, \tilde{\nabla})$ has parallel curvature and parallel torsion, i.e., $\tilde{\nabla}K = 0, \tilde{\nabla}\mathcal{T} = 0$.

4) $\tilde{\nabla}\mathcal{S} = 0, \tilde{\nabla}(\mathcal{V}\mathcal{S}) = 0, \mathcal{V}\varphi = 0$.

The next definition brings together all the algebraic compatibility conditions among the tensor fields $\varphi, \mathcal{S}, \mathcal{K}$ and $\mathcal{T}$:

Definition 2. An algebraic $\varphi$-manifold is a collection $(V, \varphi_0, \mathcal{S}_0, \mathcal{K}_0, \mathcal{T}_0)$, where $V$ is (real) vector space, $\varphi_0$ is a positive inner product on $V$, $\mathcal{S}_0, \mathcal{K}_0, \mathcal{T}_0$ are tensors of types $(4,4), (4,3), (4,2)$ respectively, and the following
Conditions are satisfied:

(i) Both $S_0$, $I - S_0$ are non-singular transformations of $Y$

(ii) For any $X, Y \in V$ the endomorphism $\tilde{X}_0(X,Y)$ acting as derivation on the tensor algebra $\mathcal{T}(V)$ satisfies

$$\tilde{X}_0(X,Y)\tilde{X}_0 = \tilde{X}_0(X,Y)\tilde{X}_0 = X_0(X,Y) = X_0(X,Y)S_0 = 0$$

(iii) The tensors $\tilde{X}_0$, $\tilde{Y}_0$, $\varphi_0$ are invariant by $S_0$

(iv) $\tilde{X}_0(X,Y) = -\tilde{X}_0(Y,X)$, $\tilde{Y}_0(X,Y) = -\tilde{Y}_0(Y,X)$

(v) The first Bianchi identity

$$\sigma(\tilde{X}_0(X,Y)Z - \tilde{X}_0(\tilde{Y}_0(X,Y),Z)) = 0$$

holds.

(vi) The second Bianchi identity $\sigma(\tilde{X}_0(\tilde{Y}_0(X,Y),Z)) = 0$ holds.

We shall make use of the following theorem by A.J. Ledger ([11]):

**Theorem A.** Let $(M, g, \{\omega^i\})$ be an $\alpha$-manifold. Then the group of all isometries of $(M, g)$ keeping the tensor field $S$ invariant is a transitive Lie group. Hence $(M, g)$ is a homogeneous Riemannian manifold $\mathcal{G}/\mathcal{H}$ and it is a complete Riemannian space.

On account of this theorem we can make

**Definition 3.** Two $\alpha$-manifolds $(M, g, \{\omega^i\}), (M', g', \{\omega'^i\})$ are called locally isomorphic if for any two points $p \in M$, $p' \in \alpha M'$ there is an isometry $\phi$ of a neighbourhood $U \ni p$ onto a neighbourhood $U' \ni p'$ such that $\phi(S|_U) = S'|_{U'}$.

**Definition 4.** Two algebraic $\alpha$-manifolds $(V_i, g_i, \tilde{X}_i, \tilde{Y}_i)$ $i = 1, 2$ will be called isomorphic if there is a linear isomorphism $\mathbf{f}: V_1 \rightarrow V_2$ of vector spaces such that $\mathbf{f}(g_1) = g_2$. 

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Theorem 3. Let \((M, \varphi, \{\alpha_x\})\) be an \(\alpha\)-manifold. Then for each point \(\varphi \in M\) the collection \((M_\varphi, \varphi_\varphi, S_\varphi, \tilde{\alpha}_\varphi, \tilde{T}_\varphi)\) is an algebraic \(\alpha\)-manifold and for any two points \(\varphi, \xi \in M\) the corresponding algebraic \(\alpha\)-manifolds are isomorphic.

We shall call the isomorphism class of all \((M_\varphi, \varphi_\varphi, S_\varphi, \tilde{\alpha}_\varphi, \tilde{T}_\varphi), \varphi \in M\) the algebraic type of the \(\alpha\)-manifold \((M, \varphi, \{\alpha_x\})\).

Theorem 4 (Equivalence theorem). Two \(\alpha\)-manifolds are locally isomorphic if and only if they have the same algebraic type.

Notice that two locally isomorphic simply connected \(\alpha\)-manifolds are globally isomorphic.

Using a construction by K. Nomizu ([4]), we can prove

Theorem 5 (Existence theorem). Any algebraic \(\alpha\)-manifold is the algebraic type of a (unique) simply connected \(\alpha\)-manifold.

We have also the following result by A.J. Ledger, which corresponds to Theorem 6.2 of [1].

Theorem B. For any \(\alpha\)-manifold there is a simply connected covering \(\alpha\)-manifold such that the covering map is a local isomorphism in the neighbourhood of each point.

The following result is useful in all kinds of classification problems:
Theorem 6. Let $(M,\varphi)$ be a simply connected g.s. space and let $M = M_0 \times M_1 \times \cdots \times M_\kappa$ be the de Rham decomposition of $(M,\varphi)$ (i.e., $M_0$ is the Euclidean part and $M_1,\ldots,M_\kappa$ are irreducible components). Then all Riemannian spaces $M_0, M_1,\ldots,M_\kappa$ are g.s. spaces. Moreover, any regular $\rho$-structure of order $\kappa$ on $(M,\varphi)$ determines a regular $\rho$-structure of order $\kappa|i$ on each $M_i$, where $\kappa|i$ for $i = 0,1,\ldots,\kappa$.

A modest classification problem.

According to Theorem 5, if we succeed in classifying all algebraic $\rho$-manifolds of a given dimension, then we can classify all simply connected $\rho$-manifolds and thus all simply connected generalized symmetric spaces of this dimension.

In the rest of this note we present a complete classification of all simply connected and irreducible g.s. spaces of dimensions 3, 4, 5 and of orders greater than 2 (we shall call these spaces briefly "exceptional" ones). It means, we leave out all symmetric spaces of E. Cartan which are well-known. In each case we shall give a representation in the form of a homogeneous Riemannian space (cf. Theorem A). As a rule, we shall describe first the underlying homogeneous manifold and then we give the family of all admissible invariant metrics in a different, more explicit form.

The details of the method and the complete proofs will appear as a special issue in the edition "Rozpravy ČSAV", Czechoslovak Academy of Sciences, Prague.
Dimension \( m = 3 \).

All exceptional spaces are of order 4 and of the following type:

As a homogeneous space, \( \mathcal{M} \) is the matrix group

\[
\begin{pmatrix}
\varepsilon^{-\frac{x}{2}} & 0 & 0 \\
0 & \varepsilon^{\frac{x}{2}} & 0 \\
0 & 0 & 1
\end{pmatrix}
\]

Also, \( \mathcal{M} \) is the space \( \mathbb{R}^3(x, y, z) \) with a Riemannian metric

\[
\varphi = \varepsilon^{2x} dx^2 + \varepsilon^{-2x} dy^2 + \lambda^2 dz^2,
\]

where \( \lambda > 0 \) is a constant.

The typical symmetry at the point \((0, 0, 0)\) is the transformation \( x' = -x, y' = x, z' = -z \).

Dimension \( m = 4 \).

All exceptional spaces are of order 3 and of the following type:

\[
\begin{pmatrix}
a & \beta & \mu \\
c & d & \nu \\
0 & 1 & 0
\end{pmatrix}
\]

where \( \det \begin{pmatrix} a & \beta \\ c & d \end{pmatrix} = 1 \).

Also, \( \mathcal{M} \) is the space \( \mathbb{R}^4(x, y, \mu, \nu) \) with a Riemann metric

\[
\varphi = \frac{(1+\nu^2)dx^2 + (1+\mu^2)dy^2 - 2\nu dx dy + \lambda^2}{1 + x^2 + \nu^2} (\lambda > 0)
\]

Each transformation \( \mu' = \cot \mu - \sin \nu, \quad x' = \cos 2t \cdot x - \sin 2t \cdot y \)
\[ v' = \sin t \cdot u + \cos t \cdot v, \quad y' = \sin 2t \cdot x + \cos 2t \cdot y \]

for \( t \neq m \pi \) is a symmetry at the point \((0,0,0,0)\) which extends to a regular \(\mathbb{S}\)-structure on \(M\).

**Dimension** \( n = 5 \).

All exceptional spaces are of order 4 or 6, and of the following 12 types:

**Type 1.**

As a homogeneous space, \( M \) is the matrix group

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & x \\
0 & 1 & 0 & y \\
0 & 0 & 1 & x \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

Also, \( M \) is the space \( \mathbb{R}^5(x,y,x,u,v) \) with a Riemann metric

\[ g = dx^2 + dy^2 + dw^2 + \lambda^2(xdu + ydv - dx)^2 \quad (\lambda > 0). \]

The typical symmetry at the point \((0, \ldots, 0)\) is the transformation \( x' = y, \quad y' = -x, \quad x' = -x, \quad u' = -v, \quad v' = u \).

**Type 2.**

\[
M \text{ is the matrix group } \begin{pmatrix}
e^{\alpha t} & 0 & 0 & 0 & x \\
e^{\alpha t} & 0 & 0 & 0 & y \\
e^{\alpha t} & 0 & 0 & x \\
0 & 0 & 0 & e^{\alpha t} & w \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

depending of two real parameters \( \lambda_1 > 0 \), \( \lambda_2 \geq 0 \).

Also, \( M \) is the space \( \mathbb{R}^5(x,y,z,w,t) \) with a Riemann metric

\[
ge = e^{-2\lambda_1 t} dx^2 + e^{2\lambda_1 t} dy^2 + e^{-2\lambda_1 t} dx^2 + e^{2\lambda_1 t} dw^2 + dt^2 + 2 \alpha [e^{(\lambda_1 + \lambda_2) t} dx + e^{(\lambda_1 + \lambda_2) t} dy + e^{(\lambda_1 + \lambda_2) t} dz] +
\]

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\[ + 2 \beta \left[ e^{(\lambda_1 - \lambda_2)^t} dy dx - e^{(\lambda_2 - \lambda_1)^t} dx dw \right] . \]

Here either \( \lambda_1 > \lambda_2 > 0 \), \( \alpha^2 + \beta^2 < 1 \), or \( \lambda_1 = \lambda_2 > 0 \), \( \alpha = 0 \), \( 0 \leq \beta < 1 \), or \( \lambda_1 > 0 \), \( \lambda_2 = 0 \), \( \alpha = 0 \), \( 0 < \beta < 1 \).

The typical symmetry at the point \((0, \ldots, 0)\) is the transformation \( x' = -y \), \( y' = x \), \( z' = w \), \( w' = x \), \( t' = -t \).

**Type 3.** \( M \) is the homogeneous space \( \text{SO}(3, \mathbb{C})/\text{SO}(2) \), where \( \text{SO}(3, \mathbb{C}) \) denotes the special complex orthogonal group and \( \text{SO}(2) \) denotes the subgroup \( \left\{ \begin{array}{cc} \text{SO}(2) & 0 \\ 0 & 1 \end{array} \right\} \) of \( \text{SO}(3, \mathbb{C}) \).

The Riemann metric \( g_\ast \) in \( M \) is induced by the following real invariant positive semi-definite form on the group \( \text{GL}(3, \mathbb{C}) \) of all regular complex matrices:

\[
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\beta_1 & \beta_2 & \beta_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
\]

\[
\tilde{\gamma} = \lambda^2 (\omega_1 \bar{\omega}_1 + \omega_2 \bar{\omega}_2) + \gamma (\omega_1^2 + \bar{\omega}_1^2 + \omega_2^2 + \bar{\omega}_2^2) + \mu^2 \left( \frac{\omega_3 - \bar{\omega}_3}{\lambda} \right)^2
\]

where \( \omega_1 = a_1 d a_3 + b_2 d b_3 + c_2 d c_3 \), \( \omega_2 = a_3 d a_1 + b_3 d b_1 + c_3 d c_1 \), \( \omega_3 = a_1 d a_2 + b_1 d b_2 + c_1 d c_2 \), and \( \lambda, \gamma, \mu \) are real parameters satisfying \( \lambda > 0 \), \( \mu > 0 \), \( |2 \gamma| < \lambda^2 \).

The typical symmetry at the origin of \( M \) is induced by the following transformation of \( \text{GL}(3, \mathbb{C}) \):

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Type 4.

$M$ is the complex matrix group depending of a complex parameter $\lambda$:

$$
\begin{pmatrix}
\alpha_1 & \alpha_2 & \alpha_3 \\
\bar{\alpha}_1 & \bar{\alpha}_2 & \bar{\alpha}_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
\begin{pmatrix}
\bar{x}_1 & \bar{x}_2 & \bar{x}_3 \\
-\bar{\alpha}_2 & -\bar{\alpha}_1 & -\bar{\alpha}_3 \\
c_2 & -c_1 & c_3
\end{pmatrix}

Here $x, \bar{x}$ denote complex variables and $t$ a real variable.

Also, $M$ is the space $\mathbb{C}^2(x, \bar{x}) \times \mathbb{R}^4(t)$ with a (real) Riemann metric

$$
q = e^{-\lambda+\bar{\lambda}t} dxd\bar{x} + e^{\lambda+\bar{\lambda}t} dwd\bar{w} + (dt)^2 + 2\mu \left[e^{\lambda+\bar{\lambda}t} dxd\bar{w} + e^{-\lambda+\bar{\lambda}t} d\bar{x}d\bar{w}\right] + \alpha_2 e^{-2\lambda t}(dx)^2 + \alpha e^{-2\lambda t}(d\bar{x})^2 - \alpha e^{2\lambda t}(dw)^2 - \alpha e^{2\lambda t}(d\bar{w})^2.
$$

Here $\lambda, \alpha$ are complex parameters, $\mu$ a real parameter, $\alpha \overline{\alpha} + \mu^2 < 1/4$. In case that $\lambda + \bar{\lambda} = 0, \mu = 0$ we have $\alpha = 0$.

The typical symmetry at the point $(0, 0; 0)$ is the transformation $x' = i\bar{w}, w' = ix, t' = -t$.

Types 5a, 5b.

$M$ is the homogeneous space $SO(3) \times SO(3)/SO(2)$, or $SO(2,1) \times SO(2,1)/SO(2)$, where $SO(2)$ denotes the subgroup

$$
\begin{pmatrix}
\cos t & -\sin t & 0 \\
\sin t & \cos t & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos t & \sin t & 0 \\
-\sin t & \cos t & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos t & -\sin t & 0 \\
\sin t & \cos t & 0 \\
0 & 0 & 1
\end{pmatrix}
$$

The Riemann metric $q$ is induced by the following real in-
variant positive semi-definite form on the group $GL(3,\mathbb{R}) \times GL(3,\mathbb{R})$ of all non-singular product matrices
\[
\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
\times
\begin{pmatrix}
\bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\
\bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\
\bar{c}_1 & \bar{c}_2 & \bar{c}_3
\end{pmatrix}
\]

\[\psi = \alpha^2 [(\omega_1 + \bar{\omega}_1)^2 + (\omega_2 + \bar{\omega}_2)^2] + \beta^2 [(\omega_3 - \bar{\omega}_3)^2 + (\bar{\omega}_3 - \omega_3)^2] + \gamma^2 (\omega_3 + \bar{\omega}_3)^2,
\]

where $\omega_1 = a_2 d_3 + b_2 d_3 \pm c_2 d_3$ and $\bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3$ are given by similar expressions in $\omega_1, \omega_2, \omega_3$.

Here $\alpha, \beta, \gamma$ are positive real parameters, $\alpha \equiv \beta$, and the $(+)$ and $(-)$ signs in $\omega_1, \omega_2, \omega_3$ correspond to the elliptic case 5a and to the hyperbolic case 5b respectively.

The typical symmetry at the origin of $M$ is induced by the following transformation of $GL(3,\mathbb{R}) \times GL(3,\mathbb{R})$:
\[
\begin{pmatrix}
a_1 & a_2 & a_3 \\
b_1 & b_2 & b_3 \\
c_1 & c_2 & c_3
\end{pmatrix}
\times
\begin{pmatrix}
\bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\
\bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\
\bar{c}_1 & \bar{c}_2 & \bar{c}_3
\end{pmatrix}
\rightarrow
\begin{pmatrix}
\bar{a}_1 & \bar{a}_2 & \bar{a}_3 \\
\bar{b}_1 & \bar{b}_2 & \bar{b}_3 \\
\bar{c}_1 & \bar{c}_2 & \bar{c}_3
\end{pmatrix}
\times
\begin{pmatrix}
a_1 & -a_2 & a_3 \\
b_1 & b_2 & -b_3 \\
c_1 & -c_2 & c_3
\end{pmatrix}
\]

Types 6a, 6b.

$M$ is the homogeneous space $SU(3) / SU(2)$, or $SU(2,1) / SU(2)$.

Also, $M$ is the submanifold of $C^3(x^1, x^2, x^3)$ given by the relation $x^1 \overline{x^1} + x^2 \overline{x^2} \pm x^3 \overline{x^3} = \pm 1$. The Riemann metric on $M$ is induced by the following hermitian metric on $C^3$:

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\[
\mathcal{G} = \lambda (dx^1dx^1 + dx^2dx^2 + dx^3dx^3) + \\
= \mu (x^1dx^1 + x^2dx^2 + x^3dx^3)(dx^1dx^1 + dx^2dx^2 + dx^3dx^3)
\]

where \(\lambda, \mu\) are real parameters such that \(\lambda > 0, \mu \neq 0\) and \(\mu \pm \lambda > 0\). The (+) and (−) signs correspond to the elliptic case 6a and to the hyperbolic 6b respectively.

The typical symmetry at the point \((0,0,4)\) of \(M\) is induced by the following transformation of \(C^3\):

\[
x_1' = x_2, \quad x_2' = -x_1, \quad x_3' = x_3.
\]

**Remark.** The case 6a was communicated to me orally by A.W. Deicke.

**Type 7.**

\[
\begin{pmatrix}
e^t & 0 & 0 & 0 & x \\
0 & e^{-t} & 0 & 0 & y \\
te^t & 0 & e^t & 0 & u \\
0 & -te^{-t} & 0 & e^{-t} & v \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\(M\) is the real matrix group \((t,x,y,u,v)\) are real variables and \(\lambda\) is a real parameter.

Also, \(M\) is the space \(\mathbb{R}^5(x,y,u,v,t)\) with a Riemann metric \(g = (dt)^2 + e^{-2t}(tdx - du)^2 + e^{-2t}(tdy + dv)^2 + \\
+ \mu(e^{-2t}dx^2 + e^{2t}dy^2) + 2\tau(dydu - dxdv),\)

where \(\lambda, \mu, \tau\) are real parameters, \(\lambda \geq 0, \mu > 0, \tau^2 < \mu\).

The typical symmetry at the point \((0,\ldots,0)\) is the transformation \(x' = -y, y' = x, u' = -v, v' = u, t' = -t\).

**Types 8a, 8b.**

\(M\) is the homogeneous space \(I^e(\mathbb{R}^3)/\mathbb{S}O(2)\) or
$I^h(\mathbb{R}^3) \!\! / S\!\! 0(2)$, where $I^e(\mathbb{R}^3)$, or $I^h(\mathbb{R}^3)$, denotes the group of all positive affine transformations of the space $\mathbb{R}^3(x,y,z)$ that preserve the differential form $dx^2 + dy^2 + dz^2$, or $dx^2 + dy^2 - dz^2$, respectively. ($I^e(\mathbb{R}^3)$ is the semidirect product of $SO(3)$ and $t(3)$ and $I^h(\mathbb{R}^3)$ is the semidirect product of $SO(2,1)$ and $t(3)$, where $t(3)$ denotes the translation group of $\mathbb{R}^3$.)

Also, $M$ is the submanifold of $\mathbb{R}^6(x,y,x;\alpha,\beta,\gamma)$ given by the relation $\alpha^2 + \beta^2 + \gamma^2 = \pm 1$. The Riemann metric on $M$ is induced by the following non-singular invariant quadratic form on $\mathbb{R}^6$:

$$\tilde{\varrho} = dx^2 + dy^2 \pm dz^2 + z^2(dx^2 + \beta^2 \pm d\gamma^2) +$$

$$+ [\mu^2 \pm (-1)](\alpha \, dx + \beta \, dy \pm \gamma \, dz)^2$$

where $\lambda > 0$, $\mu > 0$ are real parameters. The (+) and (−) signs correspond to the elliptic case $8a$ and to the hyperbolic case $8b$ respectively. In the elliptic case $\mu \neq 1$.

The typical symmetry at the point $(0,0,0;0,1)$ of $M$ is induced by the following transformation of $\mathbb{R}^6$:

$x' = -y$, $y' = x$, $z' = -z$, $\alpha' = \beta$, $\beta' = -\alpha$, $\gamma' = \gamma$.

All preceding exceptional spaces (types 1 - 8b) are of order 4.

Type 9. (Spaces of order 6.)
$M$ is the matrix group

\[
\begin{pmatrix}
-\mathbf{u} & \mathbf{v} & \mathbf{x} \\
\mathbf{0} & \mathbf{y} & \mathbf{0} \\
\mathbf{0} & \mathbf{z} & \mathbf{1}
\end{pmatrix}
\]

Also, $M$ is the space $R^5(x, y, z, w, v)$ with a Riemann metric

\[
g = a^2(du^2 + dv^2 + dw^2 + dw^2) + (b^2 + 1)(e^2 - u^2 + e^2 - u^2 + e^2 - u^2 + e^2 - u^2) + (c^2 - 2)(e^2 - 2e^2 - 2e^2 - 2e^2 - 2e^2),
\]

where $a > 0$, $b > 0$.

The typical symmetry at the point $(0, \ldots, 0)$ is the transformation $x' = y$, $y' = -z$, $z' = x$, $w' = v$, $v' = -u + v$.

To conclude, let us remark that two Riemann spaces of different types are always non-isometric and within each type, the corresponding parameters are invariants of the Riemann metric.

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