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ON THE RANGE OF NONLINEAR OPERATORS WITH LINEAR ASYMPTOTES WHICH ARE NOT INVERTIBLE

Djairo Guedes de FIGUEIREDO, Brasília

Abstract: Let $A : H \rightarrow H$ be a bounded linear self-adjoint operator in a real Hilbert space H, with a closed range and a finite dimensional null-space. Assume that there exists a sequence (λ_n) of positive real numbers in the resolvent set of A, such that $\lambda_n \rightarrow 0$. Let $N : H \rightarrow H$ be a compact mapping which is not necessarily bounded, but it could have some sublinear growth for $\|u\| \rightarrow \infty$, see inequality (5). Also assume some asymptotic condition on N with respect to the null-space of A, see condition (C). Under these hypotheses it is shown that the equation $Au + Nu = h$ has a solution; this theorem is applied to prove some results on the existence of solution for the nonlinear Dirichlet problem.

Key words: Dirichlet Problem for nonlinear elliptic equations. Compact operators, completely continuous operators. Mappings of type (M), coerciveness, perturbations of bounded linear self-adjoint operators.

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§ 1. Introduction. Recently Nečas [1] published a paper with a title like the one above, where he proved the following result.

Theorem. "Let H be a real Hilbert space, $A : H \rightarrow H$ a linear bounded self-adjoint operator, with a closed range and a finite dimensional nullspace $N(A)$. Let $N : H \rightarrow H$ be a compact (on general nonlinear) mapping such that

(1) $$\| Nu \| \leq K$$

for all $u \in H$, and a fixed constant $K > 0$. Assume that, for each $w \in N(A)$, $\|w\| = 1$, the limit

$$(2) \qquad \ell(w) = \lim_{t \to \infty} (w, N(u+tw))$$

exists uniformly with respect to bounded sets of u. Finally suppose that, for each $w \in N(A)$, $\|w\| = 1$, we have

$$(3) \qquad (w, h) < \ell(w) ,$$

where $h \in H$ is given. Then the equation

$$(4) \qquad Au + Nu = h$$

has a solution $u \in H$."

An extension of this result was obtained by Fučík, Kučera and Nečas [2], when they relaxed (1) and (2). In this note we propose to extend these results and also present a simpler technique to proving this type of results. The main idea of the proof is a sort of perturbation argument used in similar situations by the author [3], Hess [4], and surely others. Like 2 we shall handle nonlinear mappings that are not bounded. And we present three different results according to the tape of"continuity" imposed on N : compactness, weak continuity or type M. As for the linear mapping A we essentially take the same hypotheses as 1. In Section 5 we apply our results to the type of boundary value problem for semilinear elliptic equations discussed in [2].

We would like to thank Prof. L. Nirenberg for supplying us with the preprint of paper [2].

§ 2. Equation with a compact nonlinear part. A mapping $N : X \to Y$ between two normed spaces X and Y is seid to compact if (i) it is continuous in the norm topologies, and (ii) it takes bounded sets of X into relatively compact sets of Y. In this section we shall study the solvability of the equation

$$Au + Nu = h , \tag{4}$$

where h is a given element in a real Hilbert space H, and $N : H \to H$ is a compact mapping. The main result is as follows.

Theorem 1. *Let $A : H \to H$ be a bounded linear self-adjoint operator in a real Hilbert space H, with a closed range $R(A)$ and a finite dimensional null-space $N(A)$. Assume that there exists a sequence (λ_m) of positive real numbers in the resolvent set of A, such that $\lambda_m \to 0$. Let $N : H \to H$ be a compact mapping such that*

$$\| Nu \| \leq c \| u \|^{\alpha} + K , \tag{5}$$

for all $u \in H$, where $c > 0, K > 0, 0 \leq \alpha < 1$, are fixed constants. Assume that the following condition holds:

(C) *Given $y \in N(A)$, $\|y\| = 1$, and sequences $t_m \to +\infty$, $y_m \to y$, $y_m \in N(A)$, $x_m \in R(A)$, $\|x_m\| \leq K_1$, where K_1 is a constant, we have*

$$(h, y) > \liminf_{m \to \infty} (N(t_m y_m + t_m^{\alpha} x_m), y) . \tag{6}$$

Then equation (4) has a solution $u \in H$.

Remark. If we have the information that there exists a sequence of negative real numbers λ_m in the resolvent set of A, then the inequality (6) is replaced by

(6') $\qquad (h, y) < \limsup_{n\to\infty} (N(t_n y_n + t_n^{\alpha} x_n), y)$.

<u>Proof</u>: Consider first the approximant equations

$$(7) \qquad Au_n - \lambda_n u_n + Nu_n = h \ ,$$

which we prove now that it is solvable for each n . Indeed, (7) is equivalent to

$$(8) \qquad u_n = (A - \lambda_n)^{-1} (h - Nu_n) \ .$$

The mapping $T : H \longrightarrow H$ defined by $Tu = (A - \lambda_n)^{-1}(h - Nu)$ is compact and

$$\| Tu \| \le c_1 (\| h \| + \| Nu \|) - c_2 \| u \|^{\alpha} + c_3 \ ,$$

in view of (5). Thus for $\| u \| = R$, with R sufficiently large, we have $\| Tu \| \le R$. So, by a version of the Schauder fixed point theorem, T has a fixed point u_n , which is a solution of (7).

Next we claim that the sequence (u_n) is bounded. Suppose for the moment that this has been proved and let us complete the proof. In virtue of the hypotheses on A , we see that $H = N(A) \oplus R(A)$. So let us write $u_n = v_n + w_n$, where $v_n \in N(A)$ and $w_n \in R(A)$. Passing to subsequences we may assume that $v_n \to v$ and $w_n \rightharpoonup w$, where " $\longrightarrow$ " denotes convergence in the norm and " $\rightharpoonup$ " denotes weak convergence. We may also assume that $Nu_n \longrightarrow g$. So we get from (7) that $Aw_n \to h - g$. Since the mapping A restricted to $R(A)$ is a linear homeomorphism, we obtain that

$w_n \to w$. Let us denote $u = v + w$. So $u_n \to u$, and from (7) we obtain

$$Au + Nu = h ,$$

that is, u is a solution of (4).

In order to complete the proof, let us assume, by contradiction, that $\|u_n\| \to \infty$. Let us write $u_n = v_n + w_n$, where $v_n \in N(A)$ and $w_n \in R(A)$. Denoting by P the orthogonal projection of H onto $R(A)$, we obtain from (7)

$$Aw_n - \lambda_n w_n + PNu_n = Ph . \tag{9}$$

Since A restricted to $R(A)$ is a linear homeomorphism we have from (9):

$$\|w_n\| \leq c_4 [\,|\lambda_n|\,\|w_n\| + c_5 \|u_n\|^{\alpha} + K + \|h\|\,]$$

or

$$\|w_n\| \leq c_5 \|u_n\|^{\alpha} + c_6 , \tag{10}$$

for n sufficiently large. Now, let us denote $U_n = u_n / \|u_n\|$, $V_n = v_n / \|u_n\|$ and $W_n = w_n / \|u_n\|$, so that $U_n = V_n + W_n$. Going to subsequences, if necessary, we may assume in view of the finite dimensionality of $N(A)$ that $V_n \to y$, and, in view of (10), that $W_n \to 0$. So $U_n \to y$ and $\|y\| = 1$. Next, we obtain from (7) that

$$(AU_n, y) - \lambda_n (U_n, y) + \frac{1}{\|u_n\|} (Nu_n, y) = \frac{1}{\|u_n\|} (h, y) . \tag{11}$$

Since A is self-adjoint, $(AU_n, y) = (U_n, Ay) = 0$, because $y \in N(A)$. Thus from (11) it follows that

$$\lambda_n \|u_n\| (U_n, y) = (Nu_n - h, y) \quad .$$

So for n sufficiently large we have

$$(12) \qquad (Nu_n - h, y) > 0 \quad .$$

Now observe that

$$u_n = \|u_n\| V_n + w_n \ , \quad w_n = \|u_n\|^{\alpha} z_n$$

where z_n is bounded in view of (10). Thus, it follows from (12) that

$$\lim\inf (Nu_n, h) \geq (h, y) \ ,$$

which contradicts condition (C).

§ 3. Equation with a weakly continuous nonlinear part. A mapping $N: H \longrightarrow H$ in a Hilbert space H is said to be weakly continuous if $x_n \rightharpoonup x$ then implies that $Nx_n \longrightarrow Nx$.

Theorem 2. Same statement as in Theorem 1, except that the compactness of N is replaced by the assumption that N is weakly continuous.

Proof follows the same steps. The only differences are (i) The fixed point of T is guaranteed by the following well known result. "Let $T: H \longrightarrow H$ be a weakly continuous mapping such that the boundary of a ball of radius R centered at the origin is mapped into the ball. Then T has a fixed point". This is a result that can be easily proved by Galerkin approximations, i.e., projection onto finite dimensional subspaces. (ii) Once the sequence (u_m) has been proved to be bounded, we complete the proof in a simpler way. Namely, going to a subsequence, we may assume that $u_m \longrightarrow u$. Since $Au_m \longrightarrow Au$ and now $Nu_m \longrightarrow Nu$, we pass to the limit in (7), and obtain that this u is a solution of (4).

§ 4. Equation with a nonlinear part of type (M). A mapping $N : H \longrightarrow H$ in a Hilbert space H is said to be of type (M) if the following conditions hold:

(M_1) If a sequence (u_m) in H converges weakly to an element u , the sequence $Nu_m \longrightarrow w$ and $\limsup (Nu_m, u_m) \leq \leq (w, u)$, then $Nu = w$.

(M_2) N is continuous from finite dimensional subspaces of H to H endowed with its weak topology. The concept of mappings of type (M) was introduced by Brezis [5] on a more general set up. This class includes all the hemicontinuous monotone mappings and the class of pseudomonotone mappings introduced in [5]. We recall the following results, and refer to [6] for proofs.

Proposition 1. Let N be a mapping of type (M) in the Hilbert space H . Let $A : H \longrightarrow H$ be a bounded linear monotone operator. Then $A + N$ is of type (M).

Proposition 2. Let $T : H \longrightarrow H$ be a bounded mapping of type (M). Suppose that T is coercive, that

$$\lim_{\|u\| \to \infty} \frac{(Tu, u)}{\|u\|} = \infty$$

Then the range $R(T)$ of T is all of H .

Now we state and prove the main result of this section.

Theorem 3. Let $A: H \to H$ be a bounded linear monotone operator in a Hilbert space H, with a closed range $R(A)$, and a finite dimensional nullspace $N(A)$. Let $N: H \to H$ be a mapping of type (M) such that

$$\|Nu\| \leq c\|u\|^{\alpha} + K \quad , \tag{13}$$

for all $u \in H$, where $c > 0, K > 0, 0 \leq \alpha < 1$ are fixed constants. Assume that the following condition holds:

(C_M) Given $y \in N(A)$, $\|y\| = 1$, and sequences $t_n \to +\infty$, $y_n \to y$, $y_n \in N(A)$, $z_n \in R(A)$, $\|z_n\| \leq K_1$, where K_1 is a constant, we have

$$(h, y) < \lim\sup\, (N(t_n y_n + t_n^{\alpha} z_n), y) \, . \tag{14}$$

Then equation (4) is solvable in H .

Proof: We use the approximant equations

$$Au_n + \frac{1}{n} u_n + Nu_n = h \quad , \tag{15}$$

which we claim is solvable for each n. Indeed, by Proposition 1, the mapping $T = A + \frac{1}{n} I + N$ is of type (M). It follows from the boundedness of A and from (13) that T is bounded. Also from the monotonicity of A and from (13) it follows that T is coercive. So Proposition 2 may be applied, and there is a solution u_n of(15).

As in the previous theorems one has to prove that the uence (u_n) is bounded. Let us assume that this is the e and let us complete the proof. Going to a subsequence may assume that $u_n \rightharpoonup u$. So $Au_n \to Au$, and $Nu_n \to h - Au$. On the other hand,

$$(Nu_n, u_n) = (h - Au_n - \frac{1}{n} u_n, u_n) \leq$$

$$\text{5)} \qquad \leq (h, u_n) - \frac{1}{n} \|u_n\|^2 + (Au, u) - (Au_n, u) - (Au, u_n) \ ,$$

ere we have used the monotonicity of A . So

$$\limsup\, (Nu_n, u_n) \leq (h - Au, u)$$

together it allows us to use the fact that N is of type) to get $Nu = h - Au$ .That is u is a solution of (4).

Finally, the boundedness of (u_n) is proved just like Theorem 1.

§ 5. Application to boundary value problems. We shall ndicate now the application of our Theorem 1 to proving the xistence of weak solutions of the Dirichlet problem for the quation

$$\text{17)} \quad \sum_{|\alpha|,|\beta| \leq m} (-1)^{|\beta|} D^{\beta}(a_{\alpha\beta}(x) D^{\alpha} u) + \sum_{|\alpha| \leq p} (-1)^{|\alpha|} D^{\alpha}(g_{\alpha}(D^{\alpha} u)) = f \ ,$$

here f is a given function of $L^2(\Omega)$, Ω a bounded open omain in $\mathbb{R}^n$. This is exactly the problem discussed by Neas, Fučík and Kučera. Our aim in including this problem here s to illustrate the use of Theorem 1, which we believe provides a quicker proof for the existence of solutions. In a aper under preparation we are able to discuss (17) with more general nonlinear part.

Let us denote by $(\ ,\ )$ the inner product in L^2 and by $(\ ,\ )_m$ the inner product in H_0^m. For definition of H_0^m and results on the linear Dirichlet problem see, for example, Friedman [7] or Nečas [8].
A weak solution of the generalized Dirichlet problem for (17) is a function $u \in H_0^m$ such that

$$(18)\quad \sum_{|\alpha|,|\beta|\le m} (a_{\alpha\beta} D^\alpha u, D^\beta \varphi) + \sum_{|\alpha|\le p} (g_\alpha(D^\alpha u), D^\alpha \varphi) = (f, \varphi)$$

for all $\varphi \in H_0^m$.

The following assumptions are made on the linear part:

(A_1) The coefficients $a_{\alpha,\beta}$, for $|\alpha|, |\beta| \le m$, are bounded measurable real functions defined in Ω. The coefficients $a_{\alpha\beta}$, $|\alpha| = |\beta| = m$, are uniformly continuous.

(A_2) The linear operator is uniformly strongly elliptic, i.e., there is a constant $c > 0$ such that

$$(19)\quad \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^\alpha \xi^\beta \ge c\,|\xi|^{2m}$$

for $x \in \Omega$ (a.e.) and $\xi \in \mathbb{R}^n$.

Under these assumptions, we use the Riesz-Fischer representation theorem to define the operator $A: H_0^m \to H_0^m$ by

$$(20)\quad (Au, \varphi)_m = \sum_{|\alpha|,|\beta|\le m} (a_{\alpha\beta} D^\alpha u, D^\beta \varphi)$$

for all $\varphi \in H_0^m$, which is linear, bounded, self-adjoint and has a discrete spectrum. Let us assume that 0 is an eigenvalue. It is known that the nullspace of A, $N(A)$ is finite dimensional. We shall also assume a hypothesis on the unique continuation of elements in the nullspace of A :

(A_3) The only $w \in N(A)$ such that $D^{\alpha}w$, for some $|\alpha| \leq p$, vanishes on a set of positive measure is $w = 0$.

For the non linear part we assume:

(N_1) The functions $g_{\alpha} : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous, and there exist constants $0 \leq k < 1$, $K_1 \geq 0$, $K_2 \geq 0$ such that

$$(21) \qquad |g_{\alpha}(s)| \leq K_1 |s|^{k} + K_2$$

for all $s \in \mathbb{R}$ and all $|\alpha| \leq p$.

$$(N_2) \qquad 2(m - p + 1) > n \quad .$$

Under these assumptions, we use the Riesz-Fischer theorem to define the mapping $N : H_0^m \longrightarrow H_0^m$ by

$$(22) \qquad (Nu, \varphi)_m = \sum_{|\alpha| \leq p} (g_{\alpha}(D^{\alpha}u), D^{\alpha}u) \ ,$$

which is compact, and there are constants $c > 0$ and $K > 0$ such that

$$(23) \qquad \| Nu \|_m \leq c \, \| u \|_m^{k} + K$$

for all $u \in H_0^m$. The compactness of N follows from the compact embedding of H^m into H^{m-1} , and estimate (23) can be proved using Cauchy-Schwarz's and Hölder's inequalities.

Now observe that (18) is equivalent to

$$(24) \qquad (Au, \varphi)_m + (Nu, \varphi)_m = (h, \varphi)_m , \text{ for all } \varphi \in H_0^m$$

where $h \in H_0^m$ is such that $(h, \varphi)_m = (f, \varphi)$ for all $\varphi \in H_0^m$. The existence of such an h is guaranteed by the Riesz-Fischer theorem. So the generalized Dirichlet problem is equivalent to the functional equation

$$Au + Nu = h$$

in H_0^m.

Finally, we make the following assumption on the nonlinear part.

(N_3) The two limits below exist as extended real numbers, that is, in $\mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$:

$$\lim_{s \to +\infty} g_\alpha(s) = g_\alpha^+ \text{ and } \lim_{s \to -\infty} g_\alpha(s) = g_\alpha^- ,$$

with the following provisals (i) if some g_α^+ is $+\infty$ (resp. $-\infty$) then the corresponding g_α^- is $-\infty$ (resp. $+\infty$), (ii) if some g_α^+ is $+\infty$ (resp. $-\infty$) then any other g_β^+ is either finite or $+\infty$ (resp. $-\infty$).

Under assumption (N_3) we see that

$$\ell(v) = \sum_{|\alpha| \le p} g_\alpha^+ \int_{D^\alpha v > 0} D^\alpha v + g_\alpha^- \int_{D^\alpha v < 0} D^\alpha v \tag{25}$$

is defined as an extended real number, for each $v \in N(A)$.

Now state the main theorem of this section

<u>Theorem 4</u>. <u>Asssume</u> (A_1), (A_2), (A_3), (N_1), (N_2), (N_3). <u>Suppose that for each</u> $v \in N(A)$, $\|v\|_m = 1$

$$(f, v) < \ell(v) . \tag{26}$$

<u>Then the generalized Dirichlet problem (18) has a solution</u> $u \in H_0^m$.

Proof: It is enough to use Theorem 1. By the remarks made previously in this section, all the conditions of that theorem, except (C), have been checked. Observe that since 0 is an isolated eigenvalue of A, then we can obtain sequences (λ_m) in the resolvent set of A made up of either positive real numbers or negative ones. Now we check condition (C). Let $v \in N(A)$, $\|v\|_m = 1$, $v_n \to v$, $v_n \in N(A)$, $w_n \in R(A)$, $\|w_n\| \leq K_3$, $t_n \longrightarrow +\infty$. Then

$$(27)\quad (N(t_n v_n + t_n^k w_n), v)_m = \sum_{|\alpha| \leq p} \int g_\alpha (t_n D^\alpha v_n + t_n^k D^\alpha w_n) D^\alpha v$$

$$= \sum_{|\alpha| \leq p} \Big[\int_{D^\alpha v > 0} g_\alpha (t_n D^\alpha v_n + t_n^k D^\alpha w_n) D^\alpha v + \int_{D^\alpha v < 0} \text{same expression} \Big]$$

The integrands in the last term of (27) converge pointwisely a.e. to $g_\alpha^+ D^\alpha v$ in $D^\alpha v > 0$ and $g_\alpha^- D^\alpha v$ in $D^\alpha v < 0$. Here we have used (N_2) to guarantee that $D^\alpha w_n$ is bounded in the supremum norm in view of the Sobolev imbedding theorem. Now using the dominated convergence theorem, in the case of g_α^+ finite, or Fatou's lemma, in the case of a g_α^+ infinite, we obtain

$$\lim_{n \to \infty} (N(t_n v_n + t_n^k w_n), v)_m = \ell(v) .$$

So (26) implies condition (C). And the proof of Theorem 4 is complete.

Remarks. i) (26) can be replaced by

$$(f, v) > \ell(v)$$

for all $v \in N(A)$, $\|v\|_m = 1$.

ii) Theorem 4 has, as corollaries, Theorems 3.1 and

3.2 of [2] under less stringent hypotheses.

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Universidade de Brasília

Brasília,

Brasil

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