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FREE UNIFORM MEASURES

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Abstract: There is a canonical mapping from the free complete locally convex space of a uniform space into the space of uniform measures. It is proved here that a uniform measure μ is in the image of the map if and only if finite $\lim_{M\to\infty} \mu((-M)\vee f\wedge M)$ exists for each uniformly continuous function f.

Key words: Grothendieck's theorem on completeness, molecular measures, uniform measures, free uniform measures.

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Introduction. For a uniform space X there is a particularly important class of functionals on the space $U_b(X)$ of all bounded uniform functions on X. The theory of these functionals (called uniform measures) was developed by Berezanskij [1], LeCam [10] and Frolík [6],[7].

It appears that several basic results (viz. those in § 2 below) of the theory are valid in more general setting (see § 1). In § 3 I show that this general schema applies also to the space $\mathfrak{M}_F(X)$ (whose elements I call "free uniform measures" here) introduced by Berezanskij [1]. As the space $\mathfrak{M}_F(X)$ is a completion of the free locally convex space of uniform space X [12], it follows that $\mathfrak{M}_F(X)$ is a free complete locally convex space of X.

Both the space of uniform measures and the space of free uniform measures were mentioned by Buchwalter and Pupier [5] and studied in the special case of fine uniformities by several authors [2],[4],[8],[9],[11],[13],[14],[16].

In § 4 free uniform measures are described by means of uniform measures. § 4 is self-contained in the sense that no results from §§ 1 - 3 are used there.

The notations and terminology concerning topological vector spaces are those of Schaefer [15]; particularly all locally convex spaces are Hausdorff and E^* denotes the algebraic dual of E . All the vector spaces are over the field $\mathbb{R}$ of reals. Occasionally I use $\vee$ and $\wedge$ in place of *max* and *min* .

§ 1. Approximation by molecular measures

1.1. Grothendieck's theorem (dual characterization of completion). Let $\langle E, G \rangle$ be a duality and let $\mathfrak{S}$ be a saturated family covering E of $\sigma(E,G)$ -bounded sets. Denoty be G_1 the vector space of all $\mu \in E^*$ whose restrictions to each $S \in \mathfrak{S}$ are $\sigma(E,G)$ -continuous, and endow G_1 with the $\mathfrak{S}$ -topology.

Then G_1 is a complete locally convex space in which G is dense.

For the proof see Schaefer [15, IV - 6.2].

1.2. Setting. Let X be a non-empty set, $E(X)$ be a linear subspace of the space $\mathbb{R}^X$, separating points of X . Denote by $\mathcal{M}ol(X)$ the set of all formal finite real linear combinations of elements from X ; thus $\mathcal{M}ol(X)$ is the

linear space with the base $\mathcal{X}$.
The elements of $\mathcal{M}ol(\mathcal{X})$ are called molecular measures.

There is a canonical duality $\langle E(\mathcal{X}), \mathcal{M}ol(\mathcal{X})\rangle$ given by $\langle f, \Sigma \lambda_i x_i \rangle = \Sigma \lambda_i f(x_i)$ and the topology $\sigma(E(\mathcal{X}), \mathcal{M}ol(\mathcal{X}))$ is just the topology of pointwise convergence on $\mathcal{X}$.

Now consider any saturated family $\mathfrak{S}$ covering $E(\mathcal{X})$ consisting of pointwise bounded (i.e. $\sigma(E(\mathcal{X}), \mathcal{M}ol(\mathcal{X}))$ - bounded) subsets of $E(\mathcal{X})$ and denote
$\mathfrak{M}_{\mathfrak{S}}(\mathcal{X}) = \{\mu \in E(\mathcal{X})^* \mid$ for each $S \in \mathfrak{S}$ the restriction of μ to S is continuous in the topology of pointwise convergence on $\mathcal{X}\}$.
Endow $\mathfrak{M}_{\mathfrak{S}}(\mathcal{X})$ with the $\mathfrak{S}$-topology.

Grothendieck's theorem then reads as follows:

1.3. Proposition. $\mathfrak{M}_{\mathfrak{S}}(\mathcal{X})$ is a complete locally convex space in which $\mathcal{M}ol(\mathcal{X})$ is dense.

The general Ascoli theorem (see e.g. Bourbaki [3, § 2 - Th.2]) gives

1.4. The compactness criterion. A set $\mathcal{D} \subset \mathfrak{M}_{\mathfrak{S}}(\mathcal{X})$ is relatively compact if and only if (i) the restriction of $\mathcal{D}$ to any $S \in \mathfrak{S}$ is equicontinuous and (ii) the set $\mathcal{D}(f) \subset \subset R$ is bounded for each $f \in E(\mathcal{X})$.

On every set $S \in \mathfrak{S}$ the topologies $\sigma(E(\mathcal{X}), \mathcal{M}ol(\mathcal{X}))$ and $\sigma(E(\mathcal{X}), \mathfrak{M}_{\mathfrak{S}}(\mathcal{X}))$ coincide. Hence the theorem of Mackey-Arens (see Schaefer [15; IV - 3.2]) yields

1.5. Proposition. The $\mathfrak{S}$-topology on $\mathfrak{M}_{\mathfrak{S}}(\mathcal{X})$ is consistent with the duality $\langle E(\mathcal{X}), \mathfrak{M}_{\mathfrak{S}}(\mathcal{X})\rangle$ if and only if all sets in $\mathfrak{S}$ are relatively compact (in $E(\mathcal{X})$)

with respect to the topology of pointwise convergence on $\mathcal{X}$.

§ 2. Uniform measures. Given a Hausdorff uniform space X denote by $\mathcal{U}_b(\mathcal{X})$ the space of uniform (= uniformly continuous) bounded real-valued functions on $\mathcal{X}$. Consider the family $\mathrm{U.E.B.}(\mathcal{X})$ of all equiuniform (= uniformly equicontinuous) uniformly bounded subsets of $\mathrm{U}_b(\mathcal{X})$.

Thus one obtains the space $\mathfrak{M}_{\mathrm{U.E.B.}}(\mathcal{X})$, shortly $\mathfrak{M}_{\mathrm{U}}(\mathcal{X})$, whose elements are called uniform measures.

Propositions 1.3, 1.4 apply; further the closure (in R^{X}) of any $S \in \mathrm{U.E.B}$ in the topology of pointwise convergence belongs to $\mathrm{U.E.B.}$ - hence (by 1.5) dual of $\mathfrak{M}_{\mathrm{U}}(\mathcal{X})$ identifies with $\mathrm{U}_b(\mathcal{X})$. Moreover there is the following result, due to Le Cam [10] (cf. [14, Th.2]):

2.1. Theorem. The topology $\sigma(\mathfrak{M}_{\mathrm{U}}(\mathcal{X}), \mathcal{U}_b(\mathcal{X})$ and the $\mathrm{U.E.B.}$ -topology coincide on the positive cone of $\mathfrak{M}_{\mathrm{U}}(\mathcal{X})$.

§ 3. Free uniform measures. Given a Hausdorff uniform space X denote by $\mathrm{U}(\mathcal{X})$ the space of uniform real-valued functions on $\mathcal{X}$. Consider the family $\mathrm{U.E.}(\mathcal{X})$ of all equiuniform pointwise bounded subsets of $\mathrm{U}(\mathcal{X})$. Following the schema in § 1 this gives rise to the space

$\mathfrak{M}_{\mathrm{U.E.}} = \{\mu \in \mathrm{U}(\mathcal{X})^* \mid$ for each $S \in \mathrm{U.E.}$ the restriction of μ to S is continuous in the topolohy of pointwise convergence on $\mathcal{X}\}$

endowed with the topology of $\mathcal{U}.\mathcal{E}.$ -convergence. This space will be denoted $\mathcal{M}_F$ and its elements will be called <u>free uniform measures</u>.

As in § 2 the following theorem follows from 1.3 - 1.5:

3.1. <u>Theorem</u>. (a) $\mathcal{M}_F(X)$ is a complete locally convex space in which $\mathcal{M}ol(X)$ is dense.

(b) A subset $\mathcal{D}$ of $\mathcal{M}_F(X)$ is relatively compact if and only if (i) the restriction of $\mathcal{D}$ to any $S \in \mathcal{U}.\mathcal{E}.(X)$ is equicontinuous and (ii) the set $\mathcal{D}(f) \subset \mathbb{R}$ is bounded for each $f \in \mathcal{U}(X)$.

(c) (cf. [12]) The dual of $\mathcal{M}_F(X)$ is $\mathcal{U}(X)$.

The fact in (a) together with the result by Raikov [12; Th.1] implies that $\mathcal{M}_F(X)$ is the free complete locally convex space of X - this justifies the term "free"; the name "free uniform measures" was chosen as $\mathcal{M}_F$ canonically identifies with a subset of $\mathcal{M}_u$ (see § 4).

The following theorem is an analogue of 2.1.

3.2. <u>Theorem</u>. The topology $\sigma(\mathcal{M}_F(X), \mathcal{U}(X))$ and the $\mathcal{U}.\mathcal{E}.$ -topology coincide on the positive cone of $\mathcal{M}_F(X)$.

<u>Proof</u>. As the topology $\sigma(\mathcal{M}_F, \mathcal{U})$ is coarser one must prove it is finer.

Let μ_α, $\mu \in \mathcal{M}_F$ be positive and $\lim_\alpha \mu_\alpha(g) = \mu(g)$ for each $g \in \mathcal{U}(X)$. Choose any $S \in \mathcal{U}.\mathcal{E}$ and $\varepsilon > 0$. Put $f(x) = \sup\{|g(x)| \mid g \in S\}$. Then $f \in \mathcal{U}(X)$ and $\lim_{M \to +\infty} (f - (f \wedge M)) = 0$. As the set $\{f - (f \wedge M) \mid M > 0\}$ is in $\mathcal{U}.\mathcal{E}.$ there is $M_1 > 0$ such that $\mu(f - (f \wedge M_1)) < \varepsilon$.

The set $S_1 = \{(-M_1) \vee g \wedge M_1 \mid g \in S\}$ is in $\mathcal{U}.\mathcal{E}.\mathcal{B}.$ and the restrictions of μ_α and μ to $\mathcal{U}_b(X)$ are positive elements of $\mathfrak{M}_{\mathrm{U}}(X)$ (cf. § 4). Thus from 2.1 it follows that there is α_1 such that

$|\mu_\alpha(h)-\mu(h)| < \varepsilon$ for any $h \in S_1$ and any $\alpha \geq \alpha_1$,

and $|\mu_\alpha(f - f \wedge M_1) - \mu(f - f \wedge M_1)| < \varepsilon$ for any $\alpha \geq \alpha_1$.

Then for any $g \in S$ and $\alpha \geq \alpha_1$ one has

$$|\mu_\alpha(g) - \mu(g)| \leq |\mu_\alpha(g - (-M_1) \vee g \wedge M_1)| +$$

$$+ |\mu_\alpha((-M_1) \vee g \wedge M_1) - \mu((-M_1) \vee g \wedge M_1)| + |\mu(g - (-M_1) \vee g \wedge M_1| <$$

$$< \mu_\alpha(f - f \wedge M_1) + \varepsilon + \mu(f - f \wedge M_1) < 4\varepsilon . \qquad \text{Q.E.D.}$$

The following example shows the free uniform measure need not be order bounded linear form on $\mathrm{U}(X)$ (or equivalently: the space $\mathfrak{M}_F(X)$ need not be spanned by its positive cone).

3.3. Example. Let X be the real line with the usual (metric) uniformity. For $f \in \mathrm{U}(X)$ put

$$\mu(f) = \sum_{n=2}^{\infty} \frac{1}{n^2} (f(n) - f(n + \frac{1}{n})) .$$

Then $\mu \in \mathfrak{M}_F(X)$ but for the function $g \in \mathrm{U}(X)$, $g : x \longmapsto |x|$, and for any m one can find $f \in \mathrm{U}(X)$ such that

$0 \leq f \leq g$,

$f(n) = n, f(n + \frac{1}{n}) = 0$ for $2 \leq n \leq m$ and $f(x) = 0$ for $x \geq m + 1$; then $\mu(f) = \sum_{n=2}^{m} \frac{1}{n}$.

§ 4. Connection of $\mathfrak{M}_F$ with $\mathfrak{M}_{\mathrm{U}}$. Observe that for any $\mu \in \mathfrak{M}_F(X)$ its restriction to $\mathrm{U}_b(X)$ is a uniform measure $\mu_{\mathrm{U}} \in \mathfrak{M}_{\mathrm{U}}(X)$.

4.1. Proposition [1 ; 1.9] . For any Hausdorff uniform space X the canonical linear map $\{\mu \mapsto \mu_{\mathfrak{U}}\}$: $\mathcal{M}_F(X) \longrightarrow$ $\longrightarrow \mathcal{M}_{\mathfrak{U}}(X)$ is injective.

Proof [4; 4.8.2] . Suppose $\mu_{\mathfrak{U}} = 0$, i.e. $\mu(g) = 0$ for any $g \in \mathfrak{U}_{\mathcal{b}}(X)$. Choose any $f \in \mathfrak{U}(X)$: $f = \lim_{M \to +\infty} (-M) \vee f \wedge M$ pointwise and the set $\{(-M) \vee f \wedge M\}$ is on U.E. ,hence $\mu(f) = \lim_{M \to \infty} \mu((M) \vee f \wedge M) = 0$. Q.E.D.

In the theorem 4.5 below the image of the map $\{\mu \mapsto \mu_{\mathfrak{U}}\}$ is characterized. Particular cases of 4.5 were proved by Berezanskij [1; § 8] and Berruyer and Ivol [2], however, these authors deal with order bounded measures. As example 3.3 shows there are, in general, unbounded forms in $\mathcal{M}_F(X)$ - and this is where the difficulty lies. The following facts are more or less needed in the proof of 4.5.

4.2. Lemma. Given a Hausdorff uniform space X , $\mu \in \mathcal{M}_{\mathfrak{U}}(X)$, $\varepsilon > 0$. Let $\{f_\beta\}_{\beta \in B}$ be a net, $0 \leq f_\beta \in \mathfrak{U}_{\mathcal{b}}(X)$, such that $\lim_\beta f_\beta = 0$ pointwise and the set $\{f_\beta\}$ is in U.E.(X) . Suppose $|\mu(f_\beta)| > \varepsilon$ for each $\beta \in B$.

Then there exists a strictly increasing sequence $\{\beta(n)\}$ of indices $\beta(n) \in B$ such that

$$|\mu(\max\{f_{\beta(n)} \mid 1 \leq n \leq m\})| > m \cdot \frac{\varepsilon}{2} \text{ for } m = 1, 2, \ldots .$$

Proof. Observe first that given conditions imply the index set B cannot have the largest element.

Now as $|\mu(f_\beta)| > \varepsilon$ for each $\beta \in B$ so $\mu(f_\gamma) > \varepsilon$ for some subnet $\{f_\gamma\}$ of the net $\{f_\beta\}$ or $\mu(f_\gamma) < -\varepsilon$ for some subnet $\{f_\gamma\}$ of the net $\{f_\beta\}$.

Thus I can suppose without any loss of generality that $\mu(f_\beta) > \varepsilon$ for each $\beta \in B$ (and the case $\mu(f_\beta) < -\varepsilon$ then follows by the substitution $\mu \longmapsto -\mu$).

This assumption being made construct $\beta(n)$ inductively:

Choose any $\beta(1) \in B$.

If $\beta(1), \beta(2), \dots, \beta(m)$ are found such that $\mu(h_m) > m \cdot \frac{\varepsilon}{2}$ where $h_m = \max\{f_{\beta(n)} \mid 1 \leq n \leq m\}$ then $\lim_\beta (h_m \wedge f_\beta) = 0$ pointwise and the set $\{h_m \wedge f_\beta\}$ is in U.E.B .

Hence $\mu(h_m \wedge f_{\beta(m+1)}) < \frac{\varepsilon}{2}$ for some $\beta(m+1) > \beta(m)$.

Since $(h_m \wedge f_{\beta(m+1)}) + (h_m \vee f_{\beta(m+1)}) = h_m + f_{\beta(m+1)}$ this implies $\mu(h_m \vee f_{\beta(m+1)}) = \mu(h_m) + \mu(f_{\beta(m+1)}) - \mu(h_m \wedge f_{\beta(m+1)}) > m \cdot \frac{\varepsilon}{2} + \varepsilon - \frac{\varepsilon}{2} = (m+1) \cdot \frac{\varepsilon}{2}$. Q.E.D.

For $\mu \in \mathcal{M}_U(X)$ and $f \in U(X)$ say that $\int f d\mu$ exists and $\int f d\mu = k$ iff the finite $\lim_{M \to +\infty} \mu((-M) \vee f \wedge M) = k$ exists. (Of course, $\int f d\mu = \mu(f)$ for $f \in U_b(X)$.)

Warning: In spite of the notation, $f \longmapsto \int f d\mu$ need not be additive (unless it is defined for many functions $f \in U(X)$ enough - see 4.4 and 4.5) ! Nevertheless, the following result is in force:

4.3. <u>Lemma</u>. Given a uniform space X , $\mu \in \mathcal{M}_U(X)$, $f \in U_b(X)$ and $g \in U(X)$ such that $\int g d\mu$ exists.

Then $\int (f+g) d\mu$ exists and $\int (f+g) d\mu = \int f d\mu + \int g d\mu$.

Proof. For $M > 0$ put

$$k_M = (-M) \vee (f+g) \wedge M - f - (-M) \vee g \wedge M .$$

For $x \in X$ one has $\sup_M |k_M(x)| \leq |f(x)| \leq \sup_{y \in X} |f(y)|$; hence the set $\{k_M\}$ is in U.E.B..

Moreover $\lim_{M\to\infty} k_M = 0$ pointwise and so $\lim_{M\to\infty} \mu(k_M) = 0$, that is $\int (f+g)d\mu = \mu(f) + \int g\, d\mu$. Q.E.D.

In the proposition 4.4 below the set $S \in$ U.E.(X) is said to be full iff it is of the form

$S = \{f \in U(X) \mid |f(x) - f(y)| \leq \varphi(x,y)$ for any $x, y \in X$ and $|f| \leq g\}$

where $g \in U(X)$ and φ is a uniformly continuous pseudometric on X . Any set in U.E.(X) is contained in some full set.

4.4. Proposition (Monotone convergence). Given a Hausdorff uniform space X , full set $S \in$ U.E.(X) and $\mu \in \mathcal{M}_U(X)$ such that $\int g\, d\mu$ exists for any $g \in S$.

If $\{g_\alpha\}_{\alpha\in A}$ is a net such that $g_\alpha \in S$ for each $\alpha \in A$ and $g_\alpha \searrow 0$ pointwise then $\lim_\alpha \int g_\alpha\, d\mu = 0$.

Proof. Suppose there is $\varepsilon > 0$ and a subnet $\{g_\beta\}_{\beta\in B}$ of the net $\{g_\alpha\}_{\alpha\in A}$ such that $|\int g_\beta\, d\mu| > \varepsilon$ for each $\beta \in B$. As $\int g_\beta\, d\mu = \lim_{M\to\infty} \mu(g_\beta \wedge M)$ there are constants P_β such that $|\mu(g_\beta \wedge P_\beta)| > \varepsilon$ for each $\beta \in B$. For $f_\beta = g_\beta \wedge P_\beta$ pick a strictly increasing sequence $\{\beta(n)\}$ such that (see 4.2) $|\mu(h_m)| > \varepsilon \cdot \frac{m}{2}$ (where $h_m = \max\{f_{\beta(n)} \mid 1 \leq n \leq m\}$) for $m = 1, 2, \ldots$. It holds

$h_m \in S$ for $m = 1, 2, \dots$, hence there exists $h = \lim h_m \geq 0$ and $h \in S$.

I am going to show that neither $\sup_m P_{\beta(m)} < +\infty$ nor $\sup_m P_{\beta(m)} = +\infty$ is possible.

(i) $\sup_m P_{\beta(m)} < +\infty$: Then $h \in U_b(X)$ and $\{h_m\} \in$ U.E.B. ,hence $|\mu(h)| = \lim_{m\to\infty} |\mu(h_m)| = +\infty$, contradiction.

(ii) $\sup_m P_{\beta(m)} = +\infty$: for any M pick up $n(M)$ such that $P_{\beta(n(M))} \geq P_{\beta(n)}$ for $n = 1, 2, \dots, n(M)$ and $P_{\beta(n(M))} \geq M$.

Then $h \wedge P_{\beta(n(M))} = h_{n(M)}$ for any M and consequently

$$|\int h\, d\mu| = \lim_{M\to\infty} |\mu(h \wedge P_{\beta(n(M))})| = \lim_{M\to\infty} |\mu(h_{n(M)})| = +\infty ,$$

contradiction.

4.5. <u>Theorem</u>. For a Hausdorff uniform space X and $\mu \in \mathfrak{M}_U(X)$ two conditions are equivalent:

(i) there exists $\mu_1 \in \mathfrak{M}_F(X)$ such that $\mu(f) = \mu_1(f)$ for any $f \in U_b(X)$.

(ii) $\int f\, d\mu$ exists for any $f \in U(X)$.

<u>Proof</u>. The implication (i) $\Longrightarrow$ (ii) follows from the fact that for any $f \in U(X)$ the set $\{(-M) \vee f \wedge M \mid M > 0\}$ is in U.E. and so $\mu_1(f) = \lim_{M\to\infty} \mu_1((-M) \vee f \wedge M) = \int f\, d\mu$.

For the inverse, suppose (ii) holds and define $\mu_1(f) = \int f\, d\mu$ for $f \in U(X)$; it is to show that $\mu_1 \in \mathfrak{M}_F(X)$. Clearly $\mu_1(\lambda f) = \lambda \mu_1(f)$ for $\lambda \in R$ and $f \in U(X)$.

Thus two more things remain to be proved: (I) If $\{f_\alpha\}_{\alpha\in A}$ is a net such that the set $\{f_\alpha\}$ is in U.E. and $\lim_\alpha f_\alpha = 0$ pointwise then $\lim_\alpha \int f_\alpha d\mu = 0$.

(II) μ_1 is additive on $\mathcal{U}(\mathcal{X})$.

ad (I): Since for every $f \in \mathcal{U}(\mathcal{X})$ one has $\int f d\mu = \int f^+ d\mu - \int f^- d\mu$ it suffices to prove $\lim_\alpha \int f_\alpha^+ d\mu = 0$. If this were not so there would exist $\varepsilon > 0$ and a subnet $\{f_\beta^+\}_{\beta\in B}$ of the net $\{f_\alpha^+\}_{\alpha\in A}$ such that $|\int f_\beta^+ d\mu| > \varepsilon$ for each $\beta \in B$.

Hence there are constants P_β such that $|\int (f_\beta^+ \wedge P_\beta) d\mu| > \varepsilon$ for each $\beta \in B$ and Lemma 4.2 implies there is a sequence $\{h_m\}$ such that $0 \le h_m \in \mathcal{U}_{b}(\mathcal{X})$ and $|\mu(h_m)| > m \cdot \frac{\varepsilon}{2}$ for $m = 1, 2, \ldots$, $\{h_m\} \in$ U.E.$(\mathcal{X})$ and $h_m \nearrow h \in \mathcal{U}(\mathcal{X})$.

Now for $g_m = h - h_m$ one has $g_m \searrow 0$, and from Lemma 4.3 it follows that $\lim_{m\to\infty} |\mu(g_m)| = +\infty$; as the set $\{g_m\}$ belongs to U.E.$(\mathcal{X})$ (and consequently it also belongs to some full set in U.E.) this contradicts Lemma 4.4.

ad (II): Let $f, g \in \mathcal{U}(\mathcal{X})$ be arbitrary. For $M > 0$ put

$$k_M = (-M) \vee (f+g) \wedge M - (-M) \vee f \wedge M - (-M) \vee g \wedge M .$$

Then the set $\{k_m\}$ is in U.E.$(\mathcal{X})$ and $\lim_{M\to\infty} k_M = 0$ pointwise, hence $\lim_{M\to\infty} \mu(k_m) = 0$ from (I), that is $\int (f+g) d\mu = \int f d\mu + \int g d\mu$. Q.E.D.

4.6. <u>Remark</u>. $\mathcal{M}_F(\mathcal{X})$ may be treated as a subset of $\mathcal{M}_U(\mathcal{X})$, but not as a (topological) subspace. In fact, the uniform topology (= $\mathcal{U}.E.\mathcal{B}.$ -topology) and the "free" topology (= $\mathcal{U}.E.$ -topology) agree on $\mathcal{M}_F(X)$ if and only if $\mathcal{U}_{\ell}(X) = \mathcal{U}(X)$. For, if there exist $x_m \in X$, $m = 1, 2, \ldots$ and $f \in \mathcal{U}(X)$ such that $f(x_m) > m^2$, put $\mu_m = \frac{1}{m} x_m \in \mathcal{M}ol(X)$. Then $\mu_m \longrightarrow 0$ uniformly on every set in $\mathcal{U}.E.\mathcal{B}$ but $\mu_m(f)$ does not converge.

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