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Subsequential limits of fixed point sets

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Abstract: In this paper a sequence of functions \( \{T_n\} \) that map a complete metric space \((X,d)\) into itself and that converge uniformly to \(T_0 : X \to X\) is considered. If \(F(T_n)\) denotes the set of fixed points of \(T_n\) and for all \(x \notin F(T_n)\) and all \(m, n, T_m\) satisfies
\[
d(T_m, x, F(T_n)) \leq \alpha(d(x, F(T_m))) d(x, F(T_n)) + \beta(d(x, F(T_m))) d(x, T_m, x)
\]
where \(\alpha : (0, \infty) \to [0, 1)\) and \(\beta : (0, \infty) \to [0, 1)\) are monotonically decreasing functions and \(\alpha(d(x, F(T_m))) + 2\beta(d(x, F(T_m))) < 1\), then conditions are given that insure that \(F(T_0)\) is nonempty and compact. The work generalizes the result of Bruce Hillam [1] and Diaz and Metcalf [3].

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Introduction. Throughout this paper, \((X,d)\) will denote a complete metric space.

0.1. Definition. Let \((X,d)\) be a metric space. A function \(T : X \to X\) is said to be strictly contractive if there exists a constant \(\kappa, 0 < \kappa < 1\) such that
\[
d(Tx, Ty) \leq \kappa d(x, y) \quad \text{for all} \ x \text{ and } y \text{ in } X.
\]
0.2. Definition. Let \((X, d)\) be a metric space. A function \(T: X \to X\) is said to be a contraction if
\[
d(Tx, Ty) < d(x, y)\quad \text{for all } x \text{ and } y \text{ in } X \text{ with } x \neq y.
\]

0.3. Definition. Let \((X, d)\) be a metric space and \(\varepsilon > 0\). Then the sets of the form \(S_\varepsilon(x) = \{y: d(x, y) < \varepsilon\}\) are called spheres in \(X\). The sphere \(S_\varepsilon(x)\) has \(x\) for its center, and \(\varepsilon\) for a radius.

0.4. Definition. Let \(X\) be a metric space, and \(T: X \to X\) be a function. \(\text{F}(T)\) is defined to be the set of all fixed points of \(T\).

0.5. Definition. Let \((X, d)\) be a metric space and for \(m = 1, 2, 3, 4, \ldots\) let \(K_m \subseteq X\) be a sequence of non-empty sets. We define \(\mathcal{L}(\{K_m\})\) to be the set of all possible subsequential limit points of all possible sequences \(\{K_j\}\) where \(K_j \subseteq X_j\), i.e.
\[
\mathcal{L}(\{K_m\}) = \{x \in \mathcal{L}(\{K_j\}) \mid \forall \{K_j\}, \ K_j \subseteq X_j\}.
\]
In other words, \(\mathcal{L}(\{K_m\})\) is the upper limit \(L_\varepsilon X_m\). (See Kuratowski [4], chapt. 2, § 29, III).

0.6. Definition. \(H\) is defined to be the family of all functions \(\alpha: (0, \infty) \to [0, 1]\) such that \(\alpha\) is monotonically decreasing.

Bruce Parks Hillam [1] proved:

Theorem. For \(m = 1, 2, \ldots\) let \(T_m: X \to X\) be a sequence of functions each of which has at least one fixed point \(\alpha_m\). Let \(T_0: X \to X\) be a function with a unique fixed point \(\alpha_0\) such that for all \(x\) in \(X\)
\[
(1) \quad \alpha(T_0 x, \alpha_0) \leq \alpha(d(x, \alpha_0))d(x, \alpha_0), \quad \alpha \in H.
\]
Then, if \(T_m \to T_0\) uniformly on \(X\), \(\alpha_m \to \alpha_0\).
Metcalf and Diaz [3] have considered functions where $d(Tx, F(T)) < d(x, F(T))$, where $F(T)$ is the fixed point set of the function $T$.

Bruce has shown by an example that if (1) is replaced by

$$d(T_0 x, F(T_0)) \leq \alpha(d(x, F(T_0))) d(x, F(T_0))$$

then the sequence of fixed points might not converge but the subsequential limit points are fixed points.

In our present paper we extend a few theorems of Bruce [1] and a theorem of Diaz and Metcalf [3].

If for $m = 1, 2, \ldots$ there is a sequence of functions $T_m : X \to X$ such that $F(T_m)$ is nonempty and $\alpha$, $\beta \in H$ then $\alpha_m(x)$, $\beta_m(x)$ will denote the functions

$$\alpha_m(x) = \alpha(d(x, F(T_m))), \quad \beta_m(x) = \beta(d(x, F(T_m)))$$

and

$$\alpha(T_m x, F(T_m)) \leq \alpha_m(x) d(x, F(T_m)) + \beta_m(x) d(x, T_m x)$$

will be written instead of

$$d(T_m x, F(T_m)) \leq \alpha(d(x, F(T_m))) d(x, F(T_m)) +$$

$$+ \beta(d(x, F(T_m))) d(x, T_m x)$$

for each $m$. The following lemma is due to Bruce [1].

**Lemma 1.1.** For $m = 1, 2, 3, \ldots$ let $T_m : X \to X$ be a sequence of functions such that $F(T_m)$ is nonempty. Let $T_0 : X \to X$ be continuous and suppose $T_m \to T_0$ uniformly. If \{\(a_{i_d}\}\} is a sequence where $a_i \in F(T_i)$ and such that $a_{i_d} \to x_0$ then $x_0 \in F(T_0)$ and $L_s F(T_m) \subseteq F(T_0)$.

**Lemma 1.2.** For $m = 1, 2, \ldots$, let $T_m : X \to X$ be a sequence of functions such that $F(T_m)$ is nonempty.
Suppose there are functions \( \alpha \) and \( \beta \) in \( H \) such that for all \( x \in X \mid F(T_m) \),

\[
(1.2.1) \quad d(T_m x, F(T_m)) \leq \alpha_m(x) d(x, F(T_m)) + \\
\quad + \beta_m(x) d(x, T_m x) \leq \alpha_m(x) + 2\beta_m(x) < 1.
\]

Let \( T_0 : X \to X \) be a continuous function and suppose \( T_n \to T_0 \) uniformly. Then for every \( \varepsilon_0 > 0 \) there exists an integer \( I_0 \) with the property that for each \( \alpha_{I_0} \in F(T_{I_0}) \) the following hold.

(i) There exists a convergent sequence \( \{a_{i_j}\} \) with \( a_{I_0} = a_{i_1} \) and \( a_{i_j} \in F(T_{i_j}) \);

(ii) \( d(a_{i_j}, a_{i_{j+1}}) < \varepsilon_0 \) for all positive integers \( j, k \).

**Proof:** Let \( \varepsilon_0 > 0 \) be arbitrary. Set \( \varepsilon_1 = \frac{\varepsilon_0}{2} \) and choose \( \varepsilon'_1 \) such that \( \frac{\varepsilon'_1}{1 - \lambda(\varepsilon'_1)} < \varepsilon_1 \), where \( \lambda(\varepsilon'_1) = \frac{\alpha(\varepsilon'_1) + \beta(\varepsilon'_1)}{1 - \beta(\varepsilon'_1)} \).

Since \( T_n \to T_0 \) uniformly, there exists a positive integer \( N_1 \) such that for all \( j, k \geq N_1 \), \( d(T_k x, T_j x) < \varepsilon'_1 \).

Let \( I_0 = N_1 \), \( a_{I_0} \in F(T_{I_0}) \) be arbitrary and set \( a_{i_1} = a_{I_0} \).

**Claim 1.** For every \( k \geq N_1 \), \( d(a_{i_1}, F(T_k)) < \varepsilon_1 = \frac{\varepsilon_0}{2} \).

If not, then there exists a \( k_0 \geq N_1 \) such that \( d(a_{i_1}, F(T_{k_0})) \geq \varepsilon_1 \). But then

\[
d(a_{i_1}, F(T_{k_0})) \leq d(T_{i_1} a_{i_1}, T_{k_0} a_{i_1}) + d(T_{k_0} a_{i_1}, F(T_{k_0})),
\]

Now
\[
d(d(T_{\mathfrak{a}} a_{i_1}, F(T_{\mathfrak{a}}))) \\
\leq \alpha_{T_{\mathfrak{a}}} (a_{i_1}) d(a_{i_1}, F(T_{\mathfrak{a}})) + \beta_{T_{\mathfrak{a}}} (a_{i_1}) d(a_{i_1}, T_{\mathfrak{a}} a_{i_1}) \\
\leq \alpha_{T_{\mathfrak{a}}} (a_{i_1}) d(a_{i_1}, F(T_{\mathfrak{a}})) + \beta_{T_{\mathfrak{a}}} (a_{i_1}) d(a_{i_1}, F(T_{\mathfrak{a}})) \\
+ \beta_{T_{\mathfrak{a}}} (a_{i_1}) d(F(T_{\mathfrak{a}}), T_{\mathfrak{a}} (a_{i_1})) .
\]

Or
\[
d(T_{\mathfrak{a}} a_{i_1}, F(T_{\mathfrak{a}})) \leq \frac{\alpha_{T_{\mathfrak{a}}} (a_{i_1}) + \beta_{T_{\mathfrak{a}}} (a_{i_1})}{1 - \beta_{T_{\mathfrak{a}}} (a_{i_1})} d(a_{i_1}, F(T_{\mathfrak{a}})) .
\]

Therefore
\[
d(a_{i_1}, F(T_{\mathfrak{a}})) \leq \varepsilon_1' + \lambda_{T_{\mathfrak{a}}} (a_{i_1}) d(a_{i_1}, F(T_{\mathfrak{a}}))
\]
where
\[
\lambda_{T_{\mathfrak{a}}} (a_{i_1}) = \frac{\alpha_{T_{\mathfrak{a}}} (a_{i_1}) + \beta_{T_{\mathfrak{a}}} (a_{i_1})}{1 - \beta_{T_{\mathfrak{a}}} (a_{i_1})} .
\]

This, combined with the fact that \( \alpha \) and \( \beta \) are monotone decreasing, implies
\[
d(a_{i_1}, F(T_{\mathfrak{a}})) < \varepsilon_1' \leq \frac{\varepsilon_1'}{1 - \lambda_{T_{\mathfrak{a}}} (a_{i_1})} \leq \frac{\varepsilon_1'}{1 - \lambda (\varepsilon_1)} < \varepsilon_1 .
\]

which is a contradiction.

Let \( \varepsilon_2 = \frac{\varepsilon_0}{2^2} \) and choose \( \varepsilon_2' \) such that
\[
[\frac{\varepsilon_2'}{1 - \lambda (\varepsilon_2')} ] \leq \varepsilon_2 \]
and let \( N_2 \geq N_1 \) be chosen so that for all \( j, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \) \( d(T_{\mathfrak{a}} \times, T_{\mathfrak{b}}) \times) \leq \varepsilon_2' \). Let \( a_{i_2} \in F(T_{N_2}) \) where \( i_2 = N_2 \) be chosen such that
\[
d(a_{i_1}, a_{i_2}) < \varepsilon_1 \]
which is possible by Claim 1.

By an argument similar to Claim 1, \( d(a_{i_2}, F(T_{\mathfrak{a}})) < \varepsilon_2 \)
for all \( \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d} \). Suppose that for a finite increasing sequence of integers \( \{ N_{i_2} \}_{i_2=1}^{m} \)
there corresponds a sequence of points \( \{ a_{i_2} \}_{i_2=1}^{m} \) such that
(i) $a_{i_j} \in F(T_{N_j})$ where $N_j = i_j$, $j = 1, 2, \ldots, n$,

(ii) 
$$d(a_{i_j}, a_{i_{j+1}}) < \varepsilon_{i_j} = \frac{\varepsilon_0}{2^j},$$

(iii) $d(a_{i_m}, F(T_{N_j})) < \varepsilon_m = \varepsilon_0 2^m$ for all $k \geq N_m$.

Then $N_{m+1}, a_{i_{m+1}}$ are found by setting $\varepsilon_{m+1} = \frac{\varepsilon_0}{2^{m+1}}$, choosing $\varepsilon'_{m+1}$ such that \[\frac{\varepsilon'_{m+1}}{2} \leq \lambda(\varepsilon_{m+1})\] $< \varepsilon_{m+1}$.

By the uniform convergence of $\{T_n\}$ there exists a positive integer $N_{m+1} > N_m$ such that for all $k, j \geq N_{m+1}$,
$$d(T_{N_{m+1}}, T_{N_{m+1}}) < \varepsilon_{m+1}.$$

Let $i_{m+1} = N_{m+1}$. By (iii) there is an $a_{i_{m+1}}$ in $F(T_{N_{m+1}})$ such that $d(a_{i_m}, a_{i_{m+1}}) < \varepsilon_m = \frac{\varepsilon_0}{2^m}$. Also for all $j \geq N_{m+1}$, $d(a_{i_{m+1}}, F(T_{N_{m+1}})) < \varepsilon_{m+1}$.

We continue the above procedure and let $\{a_{i_{j-1}}\}$ denote the resulting subsequence.

Claim 2. \[\{a_{i_{j-1}}\}\] is a Cauchy sequence. Let $\varepsilon > 0$ be arbitrary. Let $N$ denote the positive integer such that $(\frac{\varepsilon_0}{2^{N+1}}) < \varepsilon$. Thus for all $k, j \geq N$,
$$d(a_{i_j}, a_{i_{j-t+1}}) \leq \sum_{t=0}^{k-j-1} d(a_{i_{j+1}}, a_{i_{j+1+t+1}})$$
$$< \sum_{t=0}^{k-j} (\frac{\varepsilon_0}{2^j+t}) = (\frac{\varepsilon_0}{2^{N+1}}) < \varepsilon.$$

Thus $\{a_{i_{j-1}}\}$ is a Cauchy. So Lemma 1.2 follows. Combining Lemma 1.1 and Lemma 1.2 the following fixed point theorem is obtained.
Theorem 1.3. For \( n = 1, 2, 3, \ldots \), let \( T_n : X \to X \) be a sequence of functions such that \( F(T_n) \) is nonempty. Suppose there are \( \alpha \) and \( \beta \) in \( H \) such that for all \( x \in X - F(T_m) \) (1.2.1) holds. Let \( T_0 : X \to X \) be a continuous function and suppose \( T_m \to T_0 \) uniformly, then \( L_S F(T_m) \) is nonempty. Furthermore, \( L_S F(T_m) = F(T_0) \) and \( F(T_0) = \lim_{m \to \infty} F(T_m) \).

Proof: By Lemma 1.2, there exists at least one Cauchy subsequence \( \{a_{i_j}\} \) and since \((X, d)\) is a complete metric space, \( \{a_{i_j}\} \) converges to some element of \( X \) say \( u_0 \).

By Lemma 1.1, \( u_0 \in F(T_0) \) and \( L_S F(T_m) \subseteq F(T_0) \).

To show that \( F(T_0) = L_S F(T_m) \) it suffices to show that for every \( \varepsilon > 0 \) and for arbitrary but fixed \( a_0 \in F(T_0) \), \( \exists \) a positive integer \( N \) such that for all \( n \geq N \),

\[
d(a_0, F(T_n)) < \varepsilon.
\]

Let \( \varepsilon' \) be so chosen that

\[
\frac{\varepsilon'}{1 - \lambda(\varepsilon)} < \varepsilon, \quad \lambda(\varepsilon) = \frac{\alpha(\varepsilon) + \beta(\varepsilon)}{1 - \beta(\varepsilon)}.
\]

By the uniform convergence of \( \{T_m\} \) there is a positive integer \( N' \) such that \( d(T_n x, T_0 x) < \varepsilon' \) for all \( n \geq N' \).

Claim. For all \( n \geq N' \), \( d(a_0, F(T_n)) < \varepsilon \).

If not, then there is a \( j \geq N' \) such that \( d(a_0, F(T_j)) \geq \varepsilon \).

But then

\[
d(a_0, F(T_j)) \\
\leq d(T_0 a_0, T_j a_0) + d(T_j a_0, F(T_j)) \\
< \varepsilon' + \lambda_j(a_0) d(a_0, F(T_j)).
\]

Or

\[
d(a_0, F(T_j)) \leq [\varepsilon'/1 - \lambda_j(a_0)].
\]

But \( \alpha, \beta \) are monotone decreasing, so the above implies, by
the choice of $\varepsilon', d(\alpha_0, P(T_j)) \leq \frac{\varepsilon'}{1-\lambda(\varepsilon)} < \varepsilon$, which is a contradiction. Therefore $P(T_0) \leq L_S P(T_m)$. Finally, $P(T_0)$ is the limit of $\{P(T_m)\}$. Indeed, as $\forall \varepsilon > 0$, $\exists N \forall k \geq N$, $d(\alpha_0, P(T_k)) < \varepsilon$ it follows

$$\lim_{k \to \infty} d(\alpha_0, P(T_k)) = 0$$

i.e. $\alpha_0 \in L_+ P(T_k)$. As $L_S P(T_k) \leq P(T_0)$, we have

$$L_S P(T_k) \leq P(T_0) \leq L_+ P(T_k) \leq L_S P(T_k)$$

i.e. $P(T_0) = L_+ P(T_k) = L_S P(T_k)$,

so that $P(T_0) = L P(T_k)$.

(For notation $L, L_+$ see Kuratowski [4].)

For the special case that for every integer $m$, $P(T_m) = \{\alpha_m\}$ and $\alpha, \beta \in \mathcal{K}$ are defined to be $\alpha(t) = \kappa_1$, $\beta(t) = \kappa_2$ such that $\kappa_1 + 2\kappa_2 < 1$, $T_0$ need not be continuous, which is the import of the following theorem.

**Theorem 1.4.** For $m = 1, 2, 3, \ldots$, let $T_m: X \to X$ be a sequence of functions such that $F(T_m) = \{\alpha_m\}$. Suppose there exist strictly positive $\kappa_1$ and $\kappa_2$ with $\kappa_1 + 2\kappa_2 < 1$ such that for all $x \in X - \{\alpha_m\}$ and for all $m$

$$(1.4.1) \quad d(T_m x, \alpha_m) \leq \kappa_1 d(x, \alpha_m) + \kappa_2 d(x, T_m x)$$

Then if $T_0: X \to X$ is a function such that $T_m \to T_0$ uniformly, then $F(T_0)$ is nonempty.

**Proof:** Let $\varepsilon > 0$ be arbitrary. Since $T_m \to T_0$ uniformly, there is a positive integer $N$ such that for all $j, m \geq N$, we have
Let \( x_0 \in X \) be such that \( d(x_0, a_m) < \left( \frac{1 - \lambda^2}{4 \lambda^2} \right) \varepsilon \).

Then
\[
d(a_m, a) \leq d(T_n x_0, a_m) + d(T_n x_0, T_n x_0) + d(T_n x_0, a) \\
\leq \lambda d(x_0, a_m) + d(T_n x_0, T_n x_0) + \lambda d(x_0, a) \\
\leq 2\lambda d(x_0, a_m) + \lambda d(a_m, a) + d(T_n x_0, a) \ .
\]

Hence
\[
d(a_m, a) \leq \frac{2\lambda}{1 - \lambda} \left[ \varepsilon + \frac{A}{1 - \lambda} \right] < \varepsilon .
\]

Thus \( \{ a_m \} \) is Cauchy.

Since \((X, d)\) is complete, there exists an \( a_0 \in X \) such that \( \lim_{m \to \infty} a_m = a_0 \).

Claim. \( T_0 a_0 = a_0 \). Let \( \varepsilon > 0 \) be arbitrary and let \( N'' \) be a positive integer such that for all \( j \geq N'' \),
\[
d(a_j, a_0) < \frac{\varepsilon}{3} \quad \text{and for all } x, d(T_0 x, T_0 x) < \frac{\varepsilon}{3}.
\]

Then
\[
d(a_0, T_0 a_0) \leq d(a_0, a_j) + d(a_j, T_0 a_0) + d(T_0 a_0, T_0 a_0) \\
< \frac{\varepsilon}{3} + \lambda d(a_j, a_0) + \frac{\varepsilon}{3} < \varepsilon,
\]
which implies \( a_0 = T_0 a_0 \). Thus \( a_0 \in P(T_0) \).

In Theorems 1.3 and 1.4 conditions were given that insured that the limit function \( T_0 \) has at least one fixed point.

Theorem 1.5 below gives conditions that insure that \( P(T_0) \) is compact.

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Theorem 1.5. For \( m = 1, 2, 3, \ldots \), let \( T_m : X \to X \) be a sequence of functions such that \( F(T_m) \) is nonempty and compact. Suppose there are \( \alpha, \beta \) in \( \mathcal{H} \) such that for all \( m \) and for all \( x \in X - F(T_m) \)

\[(1.5.1) \quad \alpha(T_m x, F(T_m)) \leq \alpha_m(x) d(x, F(T_m)) + \beta_m(x) d(x, T_m x) \alpha_m(x) + 2 \beta_m(x) \leq 1.\]

Let \( T_0 : X \to X \) be a continuous function and suppose that \( T_m \to T_0 \) uniformly. Then \( F(T_0) \) is nonempty and compact.

Proof: By Theorem 1.3, \( F(T_0) \) is nonempty, thus it is sufficient to show that \( F(T_0) \) is compact. Now, a set in a metric space is compact if and only if it is both complete in itself and totally bounded. Clearly, since \( T_0 \) is continuous, \( F(T_0) \) is complete in itself.

Let \( \{a_m\} \subseteq F(T_0) \) be a Cauchy sequence with \( \mu_0 \) as its limit. Thus \( \mu_0 = \lim_{m \to \infty} a_m = \lim_{m \to \infty} T_0 a_m = T_0 \mu_0 \) i.e. \( \mu_0 \in F(T_0) \). We wish to show now that \( F(T_0) \) is totally bounded. So let \( \varepsilon > 0 \) be arbitrary. Let \( \varepsilon' \) be chosen such that \( [\varepsilon/1 - \lambda(\varepsilon - \gamma)] \leq \varepsilon/\gamma \). By the uniform convergence of the \( \{T_m\} \), there exists a positive integer \( N \) such that for all \( n \geq N \), \( d(T_n x, T_0 x) < \varepsilon' \).

Claim 1. For all \( a_0 \in F(T_0) \), \( d(a_0, F(T_n)) \leq \varepsilon/\gamma \) for all \( n \geq N \). If not, then there exists \( n \geq N \) and an \( a_0 \in F(T_0) \) such that \( d(a_0, F(T_n)) \geq \varepsilon/\gamma \). Thus \( d(a_0, F(T_n)) \leq d(T_n a_0, T_0 a_0) + d(T_n a_0, F(T_n)) \leq \varepsilon/\gamma \).
which implies that
\[ d(a_0, F(T_k)) < \frac{\epsilon'}{1 - \lambda_k(a_0)} \]
But \( \alpha_k \) and \( \beta_k \) are monotone decreasing, this coupled
with the choice of \( \epsilon' \) gives
\[ d(a_0, F(T_k)) < \frac{\epsilon'}{1 - \lambda(a_0)} < \epsilon' \]
which is a contradiction.

Now from Claim 1 there follows at once:
If \( S \) is an \( \epsilon'/\gamma \) net for \( F(T_k) \), then \( S \) is an
\( 2\epsilon'/\gamma \) net for \( F(T_0) \) so that \( F(T_0) \) is totally bounded. This completes the proof.

**Theorem 1.6.** Let \( T_n : X \rightarrow X \) be a sequence of mappings with fixed point \( a_n \) for each \( n = 1, 2, \ldots \) and
let \( T_0 : X \rightarrow X \) be a strict contraction mapping with fixed point \( a_0 \). If the sequence \( \{T_n\} \) converges uniformly to \( T_0 \) and if a subsequence \( \{a_{i_\ell}\} \) of \( \{a_i\} \) converges to a
point \( x_0 \in X \) then \( x_0 = a_0 \).

**Proof:** Let \( \epsilon > 0 \). There is a positive integer \( N \)
such that \( \ell \geq N \) implies \( d(a_{i_\ell}, x_0) < \epsilon/2 \). Therefore
\[ d(a_{i_\ell}, T_0 x_0) = d(T_0 a_{i_\ell}, T_0 x_0) + d(T_0 x_0, T_0 x_0) < \epsilon \]
for all \( \ell \geq N \).
Thus \( \{a_{i_\ell}\} \) converges to \( T_0 x_0 \). Thus \( x_0 = T_0 x_0 \) and
since the fixed point \( a_0 \) of \( T_0 \) is unique, \( x_0 = a_0 \).
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References


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