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ON EXTENSIONS OF FULL EMBEDDINGS AND BINDING CATEGORIES

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Abstract: There are investigated extensions of full embeddings from a full subcategory of a given category to full embeddings of the whole category. Some results ensuring the existence of such an extension are stated. As an application, the following result is given: For any regular infinite cardinal \( \kappa \) there is a three-object category \( M_\kappa \) such that an equational class of algebras with less than \( \kappa \) -ary operations is binding if and only if \( M_\kappa \) can be fully embedded into it.

Key-words: Category, full embedding, Kan extension, binding category, equational class.

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Our basic situation is the following: \( A \) and \( C \) are categories, \( M \) is a full subcategory of \( C \), \( \chi: M \to C \) an inclusion functor and \( T: M \to A \) is a full embedding. In the first part, the construction of a functor \( L^\chi: C \to A \) extending \( T \) is recalled. The goal of this construction is the following result: If \( M \) is dense in \( C \) and co-generates \( C \), \( L^\chi \) exists and for any \( \alpha \in A \) there is a proper class of objects of \( A \) isomorphic with it, then \( L^\chi \) is a full embedding whenever a full embedding \( S: C \to A \) extending \( T \) exists (see [9]). Further, this part completes
some techniques from [9] which make it possible to show that in the result just quoted, the existence of \( S \) can be replaced by the codensity of \( M \). The second part considers the basic situation for concrete categories \( A \) and \( C \). There is introduced a new functor \( L : C \to A \) extending \( T \) and properties of \( L \) and \( L_\kappa \) are dealt with. This enables us to state results concerning extensions of full embeddings in the concrete setting, which is done in the third part. The most powerful of them, Theorems 3.5 and 3.7, are in a close connection with Theorem 1 from [10] and, like this Theorem, they are originated from some considerations of [11]. The last part is devoted to applications to binding categories. A category \( A \) is binding if the category of graphs can be fully embedded into it (see [2]). In [11], a three-object category was found, full embeddability of which into an equational class \( A \) of unary algebras makes \( A \) to be binding and the problem was put there concerning the existence of such a small testing category for equational classes of (finitary) algebras. An affirmative solution for finitary algebras was given in [10]. In this paper we show that for any infinite regular cardinal \( \kappa \) there is a three-object category \( M_\kappa \) testing any equational class of algebras with less than \( \kappa \)-ary operations. Finally, we show that under the set axiom \( \text{mon}(M) \) there is a monadic category which cannot be tested by a small category.

Concerning concepts of the category theory see [6].
§ 1. The functor $L_{*}$

We sketch the construction of functors $L_{0}, L_{1}, ..., L_{\alpha}, ..., L_{*}$ from $C$ to $A$ extending $T$ and natural transformations $\lambda_{\beta}^{\alpha}: L_{\beta} \to L_{\alpha}$ for $\beta < \alpha$ and $\lambda_{\alpha}: L_{\alpha} \to L_{*}$ inducing the identity on $K$ presented in [9]. $L_{0} = \text{Lan}_{K} T$ is a pointwise left Kan extension of $T$ along $K$, which means that $L_{0} c$ is equal to a colimit of the functor $TP$, where $P: (K \downarrow c) \to M$ is the projection of the comma category. For an isolated $\alpha$, $L_{\alpha} c$ is a colimit of a diagram having as morphisms all morphisms of $A$ with the domain in $TM$ and the codomain $L_{\alpha-1} c$. Morphisms $f, g: Tm \to L_{\alpha-1} c$ have the same domain in this diagram if and only if $L_{\alpha-1} (\lambda) f = L_{\alpha-1} (\lambda) g$ for any morphism $\lambda: c \to m$ and any $m \in M$. $\lambda_{\alpha-1, \alpha}$ is the component of the colimiting cone with the domain $L_{\alpha-1} c$. For a limit $\alpha$, $L_{\infty} c$ is a colimit of a diagram having objects $L_{\beta} c$ and morphisms $\lambda_{\beta, \beta+1}^{\alpha}$ for $\beta < \alpha$. In both cases if $L_{\alpha} c$ is isomorphic to an $L_{\beta} c$ for $\beta < \alpha$, then we choose $L_{\alpha} c = L_{\beta} c$. If for any $c \in C$ there is $\gamma$ with $L_{\gamma} c = L_{\alpha+1} c = L_{*} c$, then $L_{*}$ appears. All $\lambda$ are pointwise epi and commute together. Of course, any of constructed functors is given up to a natural isomorphism. Denote by $\mathcal{C}_{\infty}$ or $\mathcal{C}_{*}$ the class of all colimits used in the construction of $L_{\infty}$ or $L_{*}$, respectively. Put $\mathcal{C}_{\alpha} = \mathcal{C}_{\infty} - \bigcup_{\beta < \alpha} \mathcal{C}_{\beta}$ for $\alpha > 0$. $L_{\alpha}$ exists whenever $A$ has all colimits from $\mathcal{C}_{\alpha}$ and if $A$ has all colimits from $\mathcal{C}_{*}$ and is co-well-powered, then $L_{*}$ exists.

In [9], a left $M$-full functor $F: C \to A$ was defined as a functor full with respect to morphisms of $A$ do-
Lemma. Let $L^B : C \to A$ be functors extending $T$ and $\sigma : L \to S$ a pointwise mono natural transformation inducing the identity on $X$ (i.e. $\sigma X = \text{id}_X$). If $S$ is left $M$-full, then $L$ is left $M$-full, too.

For the proof it suffices to begin as in the proof of Proposition 4 from [9] and use the fact that $\sigma$ is pointwise mono.

1.2. Lemma. Let $M$ generate and cogenerate $C$ and $F : C \to A$ be a functor such that $FX$ is faithful. Then $F$ is faithful.

Proof: For any $g + g' : c \to c'$ of $C$ one can find $m$, $m' \in M$, $f : m \to c$ and $h : c' \to m'$ with $h \cdot g = h \cdot g'$. Since $FX$ is faithful, $F$ must be faithful.

1.3. Definition. Let $F : C \to A$ be a functor, $X$ a subcategory of $C$. We say that $X$ left $F$-generates $C$ if for a morphism $f : Fc \to Fc'$ of $A$, $f = F(f')$ holds for a morphism $f' : c \to c'$ of $C$ whenever for any $x \in X$ and any morphism $\alpha : x \to c$ of $C$ there exists a morphism $\alpha' : x \to c'$ of $C$ such that $F(\alpha') = fF(\alpha)$.

This definition generalizes the concept of an inductively generating subcategory $X$ of a concrete category $(C,F)$ (it means that $F : C \to \text{End}_{A}$ is a faithful functor). Dually, right $F$-generation extends projective generation.

1.4. Lemma. If $X$ left $F$-generates $C$ and $F'$ is naturally isomorphic to $F$, then $X$ left $F'$-generates $C$.

Proof is evident.
1.5. **Lemma.** Let $P : C \to A$, $G : A \to B$ be functors and $X$ a subcategory of $C$. If $X$ left $GP$-generates $C$ and $G$ is faithful, then $X$ left $P$-generates $C$.

Proof is evident.

1.6. **Lemma.** Let $L : C \to A$ be a left $M$-full functor. Then $L$ is full iff $M$ left $L$-generates $C$.

Proof is evident.

1.7. **Proposition:** Let $L : C \to A$ be a functor extending $T$ and $M$ left $L$-generate $C$. Then $L$ is right $M$-full.

**Proof:** If $c \in C$, $m' \in M$ and $f : c \to m'$ is a morphism of $C$, then the fullness of $T$ implies that $fL(\alpha) = L(\alpha')$ for any $m \in M$ and $\alpha : m \to c$. Therefore $f = L(f')$.

A family $\{f_i\}$ of morphisms of $A$ with the same codomain $\alpha$ is called jointly epi whenever $f = g_i$ for any parallel pair of morphisms of $A$ with the domain $\alpha$ such that $ff_i = gff_i$ for any $i$.

1.8. **Proposition:** Let $X$ be dense in $C$, $I : X \to C$ the inclusion, $F : C \to A$ a faithful functor and $G : A \to B$ a jointly epi family in the full subcategory of $A$ determined by $FC$ for any $c \in C$. Then $X$ left $F$-generates $C$.

**Proof:** Following the first part of the proof of Proposition 3 from [9] it suffices to take for $f$ from 1.3 the natural transformation $\tau : C(I-, c) \to C(I-, c')$ defined by $F(\tau_{\alpha}(\lambda)) = fF(\lambda)$, to use the density of $X$ for finding $f' : c \to c'$ in $C$ with $F(f'\alpha) = fF(\alpha)$ for any.
\[ \lambda : m \to c, m \in M \] and to realize that \( F(f') = f \) for \( \{F(\lambda)\} \) is jointly epi.

1.9. Corollary. Let \( M \) be dense in \( C \) and \( L : C \to A \) a faithful functor extending \( T \) such that the unique natural transformation \( \sigma : L_0 \to L \) inducing the identity on \( K \) is pointwise epi. Then \( M \) left \( L \)-generates \( C \).

Proof: In the same way as in the last six lines of the proof of Proposition 3 from [9] it can be proved that \( \{L(\lambda)/\lambda : m \to c, m \in M\} \) is a jointly epi family in \( A \) for any \( c \in C \). Hence \( M \) left \( L \)-generates \( C \) by 1.8.

Since 1.2 holds and \( \lambda_0, \lambda \) are pointwise epi, functors \( L_0, ..., L_{\lambda}, ..., L_M \) fulfill suppositions of 1.9 whenever \( M \) cogenerates \( C \).

1.10. Corollary. Let \( M \) be dense in \( C, F: C \to A \) a faithful and right \( M \)-full functor and \( F_M \) cogenerate the full subcategory of \( A \) determined by \( F \). Then \( M \) left \( L \)-generates \( C \).

Proof: Let \( f = g : Fc \to Fc' \) such that \( fF(\lambda) = gF(\lambda) \) for any \( \lambda : m \to c, m \in M \). There is \( m' \in M \) and \( \lambda : Fc' \to Fm' \) \( \lambda F = \lambda g \). Since \( F \) is right \( M \)-full, \( \lambda F = \lambda F(\lambda_1) = \lambda F(\lambda_2) \). There is \( m \in M \) and \( \lambda : m \to c \) with \( \lambda_1 \lambda = \lambda_2 \lambda \) because \( M \) generates \( C \). Hence \( \lambda F(\lambda) = F(\lambda_1 \lambda) \mp F(\lambda_2 \lambda) = \lambda g F(\lambda) \), which is a contradiction.

1.11. Theorem. Let \( M \) be dense and codense in \( C, L_M \) exist and let any \( \omega \in A \) have a proper class of objects of \( A \) isomorphic with it. Then \( L_M \) is a full embedding.

Proof: \( L_M \) is faithful by 1.2 and right \( M \)-full by 1.9 and 1.7. Further, \( \{L_M(\lambda)/\lambda : c \to m, m \in M\} \) is jointly
mon by the construction of $L_*^c$ for any $c \in C$. The dual of 1.8 ensures that $M$ right $L_*^c$ -generates $C$. Now, $L_*^c$ is full by the dual of 1.6. Since $A$ has enough isomorphic copies of each of its object, $L_*^c$ can be chosen to be an embedding (the big axiom of choice is used here).

§ 2. Concrete setting

We add to our basic situation that $(C, U)$ and $(A, V)$ are concrete categories. We shall define a functor $L$ which turns out to be useful.

Let $c \in C$ and $\sim$ be the equivalence on $VL_0 c$ defined as follows: $x \sim y$ for $x, y \in VL_0 c$ iff $VL_0 (\mathcal{K}) (x) = VL_0 (\mathcal{K}) (y)$ for any $\mathcal{K} : c \rightarrow m$ in $C$ and any $m \in M$. Let $L_c : L_0 c \rightarrow L c$ be a morphism of $A$ with the following property: $V(L_c)(x) = V(L_c)(y)$ for any $x, y \in VL_0 c$ with $x \sim y$ and if $f : L_0 c \rightarrow a$ is a morphism of $A$ with $V(f)(x) = V(f)(y)$ for any such $x, y$, then there is a unique $\mathcal{K} : L_0 c \rightarrow a$ in $A$ such that $\mathcal{K} L_c = f$. Let $q : c \rightarrow d$ be a morphism of $C$ and $x, y \in VL_0 c$ such that $x \sim y$. Then $VL_0 (q)(x) \sim VL_0 (q)(y)$ and let $L(q)$ be a unique morphism of $A$ such that $L(q) L_c = L(q) L_0$. Then $L : C \rightarrow A$ is a functor and $L(L_0) \rightarrow L$ a natural transformation. Moreover, $L$ is an extension of $T$ and $L$ induces the identity on $X$ because $x, y \in VL_0 m, m \in M$ and $x \sim y$ implies that $x = VL_0 (id m)(x) = VL_0 (id m)(y) = y$. Again, $L$ is defined up to a natural isomorphism.

2.1. Definition. Let $D : S \rightarrow A$, $F : A \rightarrow X$ be functors and $a \in A$ a colimit of $D$ with the colimiting cone
We say that $F$ compresses the colimit $\alpha$ if $\{F(\tau_\phi) \rightarrow \mathcal{S}\}$ is jointly epi. Of course, if $F$ preserves colimits, then it compresses them.

2.2. Lemma. Let $D, F, \alpha, \tau$ be as in the definition and, in addition, let $FD$ have a colimit $x \in X$. Then $F$ compresses $\alpha$ iff the induced morphism $\&: x \rightarrow Fa$ is epi.

Proof is evident.

We say that an object $e$ of $C$ is nearly $1$-free whenever there exists a natural monomorphism $\mu: U \rightarrow C(e, -)$.

We recall that for a $1$-free object $\mu$ is requested to be iso. Examples of nearly $1$-free objects which are not $1$-free are supplied by concrete categories with constants because any object $e$ of such a category is nearly $1$-free with $\mu_c$ assigning to each $x \in UC$ the constant morphism with the value $x$.

2.3. Lemma. Any nearly $1$-free object is a generator of $C$.

Proof: Let $f, g: c \rightarrow d$ be morphisms of $C$ and $f \mu = g \mu$ for any $\mu: e \rightarrow c$. Hence $\mu_d U(f) = C(e, f) \mu_c = C(e, g) \mu_c = \mu_d U(g)$ and $f = g$ because $U$ is faithful and $\mu$ mono.

2.4. Proposition: Let $X$ be a subcategory of $C$ containing a nearly $1$-free object $e$ of $C$. If $X$ inductively generates $C$, then $X$ is dense in $C$. If $\nu_{Uc}(\alpha) = \nu(\mu_c(\alpha)), \alpha \in UC, c \in C$ are components of a natural
transformation \( \nu : J \rightarrow \text{Ens}(\mathbb{E}, J^-) \) where \( J : UC \rightarrow \text{Ens} \) is the inclusion, then the converse implication holds.

**Proof:** Let \( X \) inductively generate \( C \), \( I : X \rightarrow C \) be the inclusion, \( c, d \in C \) and \( \tau : C(I-, c) \rightarrow C(I-, d) \) a natural transformation. Let \( x_i \in X \), \( \varphi_i : x_i \rightarrow c \) in \( C \) and \( y_i \in \mathcal{U}x_i \) for \( i = 1, 2 \) such that

\[
(1) \quad \mathcal{U}(\varphi_i)(y_i) = \mathcal{U}(\varphi_2)(z_2) .
\]

Since \( \mu \) is natural, for \( i = 1, 2 \) it holds

\[
(2) \quad C(e, \varphi_i)\mu_{x_i} = \mu_{c}\mathcal{U}(\varphi_i) ,
\]

\[
(3) \quad C(e, \tau_{x_i}(\varphi_i))\mu_{x_i} = \mu_{d}\mathcal{U}(\tau_{x_i}(\varphi_i)) .
\]

Similarly, the naturality of \( \tau \) implies

\[
(4) \quad \tau_{c}C(\tau_{x_i}(\varphi_i), c) = C(\tau_{x_i}(\varphi_i), d)\tau_{x_i} .
\]

Evaluating (2) at \( x_i \) and using (1), we get \( \varphi_1(\mu_{x_i}(y_i)) = \varphi_2(\mu_{x_2}(z_2)) \). Further, by evaluation of (4) at \( \varphi_2 \) we obtain \( \tau_{x_1}(\varphi_1)\mu_{x_1}(y_1) = \tau_{x_2}(\varphi_2)\mu_{x_2}(y_2) \). Finally, the evaluation of (3) at \( x_i \) yields \( \mu_d(\mathcal{U}(\tau_{x_i}(\varphi_i))(y_i)) = \mu_d(\mathcal{U}(\tau_{x_2}(\varphi_2))(z_2)) \), and thus \( \mathcal{U}(\tau_{x_i}(\varphi_i))(y_i) = \mathcal{U}(\tau_{x_2}(\varphi_2))(z_2) \) because \( \mu \) is mono.

This observation makes it possible to find a mapping \( f : UC \rightarrow Ud \) such that \( f\mathcal{U}(x) = \mathcal{U}(\tau_x(x)) \) for any \( x \in X \) and \( \varphi : x \rightarrow c \). Since \( X \) inductively generates \( C \), there is a morphism \( f' : c \rightarrow d \) of \( C \) with \( \mathcal{U}(f') = f \). Therefore \( \tau = C(I-, f') \) and the density of \( X \) is verified.

The rest of the proof follows from 1.8 using the computation of the proof of 2.3 applying to \( \nu \) instead of \( \mu \).
\( \ast \) is natural if \( e \) is a-free or if \( \mu \) is defined by constants. Thus 2.4 generalizes Lemma 3 from [9].

2.5. Proposition: Let \( A \) have colimits from \( ^L_1 \) and \( TM \) contain a nearly a-free object \( e \) of \( L_0 c \). Then \( L = L_1 \).

In addition, if \( V \) compresses colimits from \( D_1 \), then \( L_x = L_1 \).

Proof: Let \( c \in C, x, y \in VL_0 c \) and \( x \sim y \). Since

\[
VL_0(\lambda)(x) = VL_0(\lambda)(y)
\]

for any \( \lambda : c \to m, m \in M \) and \( \mu \) is natural, \( L_0(\lambda)\mu_{L_0 c}(x) = \mu_{L_0}(VL_0(\lambda)(x)) = L_0(\lambda)\mu_{L_0 c}(y) \) for any \( \lambda \). Therefore

\[
\lambda^0_0(\lambda)(x) = \lambda^0_0(\lambda)(y)
\]

and further \( V(\lambda^0_0)(x) = V(\lambda^0_0)(y) \) because \( \mu \) is natural and mono (and therefore pointwise mono because \( Emb \) is complete).

Let \( f : L_0 c \to a \) be a morphism of \( A \) with \( V(f)(x) = V(f)(y) \) for any \( x, y \in VL_0 c \) and \( x \sim y \). Let \( m \in M \) and \( f', g' : M \to L_0 c \) be morphisms of \( A \) such that \( L_0(\lambda)f' = L_0(\lambda)g' \) for any \( \lambda : c \to m, m \in M \). Hence \( V(f')(x) \sim V(g')(x) \) for any \( x \in TM \) and thus \( f'x = g'x \). By the definition of \( \lambda^0_0 \), there is a unique \( \lambda : L_1 c \to a \) such that \( \lambda \lambda^0_0 = f \). Thus, \( \lambda^0_0 = \lambda \) and \( L = L_1 \).

If \( V \) compresses colimits from \( D_1 \), then \( V\lambda^0_0 \) is epi. Let \( m \in M \) and \( f, g : M \to L_1 c \) such that \( L_1(\lambda)f = L_1(\lambda)g \) for any \( \lambda : c \to m, m \in M \). For any \( x \in VTM \), there are \( x, y \in VL_0 c \) such that \( V(\lambda^0_1)(x) = V(f)(x) \) and \( V(\lambda^0_1)(y) = V(g)(y) \). The naturality of \( \lambda^0_1 \) implies that \( x \sim y \) and therefore \( V(\lambda^0_1)(x) = V(\lambda^0_1)(y) \). Thus \( f = g \).
and \( L_\# = L_\alpha \).

A close connection between \( L_\# \) and \( L \) is in a general case, too, because \( L : C \rightarrow A \) can be obtained as a restriction of \( L_\# : C_\# \rightarrow A_\# \), where \( C_\# \) and \( A_\# \) arise from \( C \) and \( A \) by a suitable addition of a \( \# \)-free object \( e \).

2.6. Proposition: Let \( TM \) contain a nearly \( \# \)-free object \( e \) of \( L_0 C \) and \( V \) compress colimits from \( C_\# \) and \( Y \) compress colimits from \( D_\# \). Then \( L_\# = L_\alpha \).

Proof follows from similar computations as in the proof of 2.5.

2.7. Lemma. Let \( TM \) contain a nearly \( \# \)-free object \( e \) of \( L_0 C \) and \( V \) compress colimits from \( C_\# \). Then any natural transformation \( \sigma : L_0 \rightarrow S \) inducing the identity on \( K \) into a left \{ \# \}-faithful functor \( S \) is pointwise mono.

Proof: Let \( c \in C \) and \( x_1, x_2 \in VL_0 C \) with \( V(\sigma_c)(x_1) = \sigma_c(x_1) \). Since \( V \) compresses colimits from \( C_\# \), there are \( m_1, m_2 \in \mathcal{N}, \xi_1 : m_1 \rightarrow c \) and \( \xi_2 \in VTm_2 \) with \( VL_0(\xi_1, x_1) = m_2(x_2) \). Since \( T \) is full, there exist morphisms \( x_2 \) of \( M \) domaining in \( e \) with \( T(x_2) = \mu_{\pm m_2}(x_2) \) for \( i = 1, 2 \). It holds \( S(\xi_1, x_1) = \sigma_c L_0(\xi_1, x_1) = \sigma_c(\mu_{\pm m_2}(x_2)) = \mu_{\pm m_2}(x_2) \) by the naturality of \( \sigma \) and \( \alpha \). The left \{ \# \}-faithfulness of \( S \) yields \( \xi_1, x_1 = \xi_2, x_2 \). Further, \( \mu_{\pm m_2}(x_1) = \mu_{\pm m_2}(VL_0(\xi_1, x_1)) = L_0(\xi_1, x_1) = L_0(\xi_1, m_2(x_2)) = L_0(\xi_1, x_2) = \mu_{\pm m_2}(x_2) \). Since \( \mu \) is mono, \( x_1 = x_2 \) and therefore \( \sigma \) is pointwise mono.

2.8. Theorem. Let \( TM \) contain a nearly \( \# \)-free object
of \( L_0 \mathcal{C} \), let \( A \) have and \( V \) compress colimits from \( \mathcal{C}_0 \). Let \( M \) be dense in \( \mathcal{C} \), any \( a \in A \) have a proper class of objects of \( A \) isomorphic with it and let there exist a full embedding \( S \) extending \( T \). Then \( L_0 \) is a full embedding.

**Proof:** \( L_0 \) is faithful by Proposition 1 from [9], and left \( M \)-full by 2.7 and 1.1. Therefore \( L_0 \) is full by 1.9 and 1.6.

This result generalizes Proposition 4 from [9].

2.9. **Definition.** We call a functor \( L: \mathcal{C} \to A \) \( V \)-covered (with respect to \( M \)) if the family \( \{VL(\xi) / m \in M, f: m \to c \} \) is jointly epi.

2.10. **Lemma.** Let \( A \) have and \( V \) compress colimits from \( \mathcal{C}_\infty \) or \( \mathcal{C}_* \). Then \( L_\infty \) or \( L_* \) resp. exists and is \( V \)-covered.

**Proof:** If \( V \) compresses colimits from \( \mathcal{C}_0 \), then \( V(\lambda^{\text{ord}}_c) \) is epi for any ordinal \( \omega \) and any \( c \in \mathcal{C} \). Thus there is only a set of \( \lambda^{\text{ord}}_c \) and therefore \( L_* \) exists. Further, \( \{VL_0(\xi) / f: m \to c, m \in M \} \) is jointly epi because \( V \) compresses the colimit \( L_0 \mathcal{C} = \text{Colim}((\mathcal{C} \downarrow c) \xrightarrow{F} M \xrightarrow{T} A) \) having \( \{L_0(\xi) / f: m \to c, m \in M \} \) as components of the colimiting cone for any \( c \in \mathcal{C} \). Since \( V \) compresses colimits from \( \mathcal{C}_\infty \) (\( \mathcal{C}_* \)) and any \( \lambda \) is a natural transformation, the assertion holds.

If \( \xi_i: \alpha_i \to a \) is a family of monics of \( A \) having a limit in \( A \), then a component of the limiting cone with the codomain \( a \) is monic which is called an intersection of \( \xi_i \).

2.11. **Lemma.** Let \( A \) have and \( V \) preserve finite limits
and arbitrary intersections. If $A$ has coequalizers and colimits from $\mathfrak{C}_0$, then $\overline{I}$ exists and if $V$ compresses them, then $\overline{I}$ is $V$-covered.

Proof: Let $c \in C$ and $i : \sim \rightarrow V L_0 c \times V L_0 c$ be the monic defined by the equivalence $\sim$. Let $d : \& \rightarrow L_0 c \times L_0 c$ be the intersection of all monics $j : \times \rightarrow L_0 c \times L_0 c$ of $A$ such that $i$ can be factorized through $V d$. Since $V$ preserves intersections, $i = V(d) \mathfrak{K}_1$ for a unique mapping $\mathfrak{K}_1$. Let $\nu_1, \nu_2 : L_0 c \times L_0 c$ be projections and $\nu$ a coequalizer of $\nu_1 d$ and $\nu_2 d$. It holds $V(\nu)V(\nu_1)i = V(\nu)V(\nu_2)i$. Let $f : L_0 c \rightarrow a$ be a morphism of $A$ with $V(\nu)V(\nu_1)i = V(\nu)V(\nu_2)i$ and $g$ an equalizer of $\nu_1 d$ and $\nu_2 d$. Since $V$ preserves finite limits, $V(g)$ is an equalizer of $V(\nu_1 d)$ and $V(\nu_2 d)$. Since $V(\nu_1 d)\mathfrak{K}_1 = V(\nu_2 d)i = V(\nu_2 d)i = V(\nu_2 d)\mathfrak{K}_1$, there is a unique $\mathfrak{K}_2$ with $V(g)\mathfrak{K}_2 = \mathfrak{K}_1$. Hence $V(dg)\mathfrak{K}_2 = i$ and the definition of $d$ yields that $V(g)$ is iso. Thus $V(\nu_1 d) = V(\nu_2 d)$ and $\nu_1 d = \nu_2 d$. There is a unique morphism $\mathfrak{K}$ of $A$ such that $\mathfrak{K}\nu_1 = f$. Hence $\overline{X}_c = \mathfrak{K}$.

Thus $\overline{I}$ exists, when $A$ has coequalizers and colimits from $\mathfrak{C}_0$. If $V$ compresses them, then $V(\overline{X}_c)$ is epi for any $c \in C$ and $L_0$ is $V$-covered by 2.10 and therefore $\overline{I}$ is $V$-covered.

Conversely, if $\overline{I}$ exists, $A$ has and $V$ preserved kernel pairs, then $\overline{X}_c$ is a coequalizer for any $c \in C$.

2.12. Lemma. Let $\overline{I}$ be $V$-covered. Then $(V L(\nu))/\mathfrak{K} : c \rightarrow m, m \in M$ is jointly monic for any $c \in C$. 

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Proof: Using the naturality of $\mathcal{A}$ it can be proved that $\forall \mathcal{A}$ is epi for $\mathcal{L}$ is $\mathcal{V}$-covered and then, by the definition of $\mathcal{L}$, that Lemma holds.

§ 3. Fulness in concrete setting

3.1. Proposition: Let $L: \mathcal{C} \rightarrow \mathcal{A}$ be a faithful functor extending $T$ such that $\forall L \cong U$ and $M$ inductively generate $\mathcal{C}$. Then $L$ is right $M$-full.

Proof follows from 1.4, 1.5 and 1.7.

$(A,Y)$ has the property of transfer if for every object $a$ of $\mathcal{A}$ and for every bijection $f$ from $\forall a$ onto an arbitrary set $x$ there is an isomorphism $f'$ of $\mathcal{A}$ such that $V(f') = f.(C,U)$ has the property of unicity, if every isomorphism $f$ of $\mathcal{C}$ such that $V(f) = \text{id}$ is an identity (see [8]). A full embedding $L: \mathcal{C} \rightarrow \mathcal{A}$ such that $\forall L = U$ is called a realization (see [7]).

3.2. Theorem. Let $(C,U)$ have the property of unicity, $(A,Y)$ of transfer, $M$ inductively and projectively generate $\mathcal{C}$. Let $L: \mathcal{C} \rightarrow \mathcal{A}$ be a faithful functor extending $T$ and let there exist a natural isomorphism between $U$ and $\forall L$ inducting an identity on $K$. Then there is a realization $\mathcal{C} \rightarrow \mathcal{A}$ extending $T$ naturally isomorphic to $L$.

Proof: $L$ is full by 1.4, 1.5, the dual of 3.1 and by 1.6. The rest of the proof follows from [8], Lemma 1.5.

3.3. Theorem. Let $(C,U)$ have the property of unicity, $(A,Y)$ of transfer, $\mathcal{A}$ have and $\mathcal{V}$ compress colimits from $\mathcal{C}_f$. Let $M$ inductively and projectively generate $\mathcal{C}$, coge-
nerate $C$ and contain a $4$-free object $e$ of $C$ such that $T e$ is a $4$-free object of $A$. Let $\mu : U \to C(\varepsilon, -)$, $\mu : V \to A(T e, -)$ be corresponding natural isomorphisms, $\beta : C(\varepsilon, -) \to A(T e, L_0 -)$ a natural transformation defined by $\beta_c(f) = L_0(f)$ and $(\mu^{-1}, L_0 \beta, \mu^0)$ be an identity.

Then $L_0$ is a realization.

Proof: Since $M$ cogenerates $C$, $L_0$ is faithful by 2.3 and 1.2 and therefore $\beta$ is mono. Hence the composition $\alpha = \mu^{-1}, L_0 \beta, \mu : U \to VL_0$ is a natural monomorphism. $L_0 \beta^0$ is iso because $T$ is full and $(VL_0(f)/m \in M; f : m \to c)$ is jointly epi by 2.10. Therefore $\alpha$ is epi. Hence $\alpha$ is a natural isomorphism and $L_0$ can be chosen as a realization by 3.2.

3.4. Theorem. Let $(C, U)$ have the property of unicity, $(A, V)$ of transfer, $M$ inductively and projectively generate $C$, $\{U(\lambda)/\lambda : c \to m, m \in M\}$ be jointly monic and $\{U(f)/f : m \to c, m \in M\}$ jointly epi for any $c \in C$. Let $T$ be a realization and $L : C \to A$ a $V$-covered functor extending $T$ such that $\{VL(\lambda)/\lambda : c \to m, m \in M\}$ is jointly monic for any $c \in C$.

Then there is a realization $C \to A$ extending $T$ naturally isomorphic to $L$.

Proof: Since $\{U(\lambda)/\lambda : c \to m, m \in M\}$ or $\{U(f)/f : m \to c, m \in M\}$ is jointly monic or epi, $M$ cogenerates or generates $C$, respectively. Hence $L$ is faithful by 1.2.

Let $c \in C$ and define a mapping $\alpha_0 : Uc \to VLc$ as follows:

$\alpha_0(x) = VL(f)(x)$ if $x = U(f)(x)$ for $f : m \to c, m \in M$.
Indeed, \( \alpha_c \) is a mapping because any \( x \) is equal to a \( U(\xi')(x) \) and \( U(\xi'_1)(x_1) = U(\xi'_2)(x_2) \) for \( \xi'_1, \xi'_2 \in M \), \( x_1 \in m_1 \) implies that \( VL(\xi_1)(x_1) = VL(\xi_2)(x_2) \) for any \( \xi_1 : c \rightarrow m, m \in M \) for \( T \) is a realization and therefore \( VL(\xi_1)(x_1) = VL(\xi_2)(x_2) \) because \( VL(\xi)/\xi : c \rightarrow m, m \in M \) is jointly monic. Let \( x_1 \neq x_2 \in Uc, x'_1 = U(\xi'_1)(x_1) \). Since \( U(\xi)/\xi : c \rightarrow m, m \in M \) is jointly monic, there is \( \eta : c \rightarrow m \) with \( VL(\eta \xi')(x'_1) = U(\eta \xi'_2)(x'_2) = VL(\eta \xi'_2)(x'_2) \). Hence \( \alpha_c \) is mono. Since \( L \) is \( V \)-covered, \( \alpha_c \) is epi. It is easy to compute that \( \alpha_c \) are components of a natural isomorphism \( \alpha : U \rightarrow VL \). Thus the Theorem follows from 3.2.

A full embedding \( L : C \rightarrow A \) is called a pseudorealization in [8] when \( Uc \subseteq VLc \) for any \( c \in C \) and these inclusions are components of a natural transformation \( U \rightarrow VL \).

3.5. **Theorem.** Let the suppositions from the first sequence of 3.4 hold. Let \( T \) be a pseudorealization, \( T \) exist and be \( V \)-covered. Let for any \( m \in M, c \in C - M \) and \( g : m \rightarrow c \) in \( C \) there exist \( n \in M \) and \( \nu_0 : c \rightarrow n \) with the following properties:

a) for any \( y \in Um - U(\nu \nu_0 q)(Um) \) there are \( s, s' : n \rightarrow n \) such that \( s \nu_0 q = s' \nu_0 q, U(s)(q) = q \) and \( U(s)(q) \neq q; \)
b) for any \( m' \in M \) and \( \mu : c \rightarrow m' (U(t)/t : m' \rightarrow n, t \nu_0 q = t \nu_0 q \) for a \( t : m \rightarrow m' \) is jointly monic.

Then \( \overline{T} \) is a pseudorealization.

**Proof:** Put \( Wa = \{ x \in V a / x = V(\xi)(x), m \in M, \xi : Tm \rightarrow a \) and \( x \in Um \} \) and \( W(q) = V(q)/Wa \) for any \( q : a \rightarrow \xi \) in
Evidently \( W : A \to \text{Emb} \) is a functor and \( WT = U \) because \( T \) is a pseudorealization. Let \( A' \) be a full subcategory of \( A \) consisting of all objects of \( A \) isomorphic with an object from \( \overline{LC} \) and \( W' \) be the restriction of \( W \) onto \( A' \). For proving that \( W' \) is faithful it suffices to show that \( W' \) is faithful on \( \overline{LC} \). Let \( c, d \in C \) and \( \varphi_1 \neq \varphi_2 : \overline{Lc} \to \overline{Ld} \). Since \( L \) is \( V \)-covered, there exists \( m \in m \) and \( f : m \to c \) such that \( \varphi_2 L(f) \neq \varphi_2 L(f) \). By 2.12 there is \( m' \in M \) and \( h : d \to m' \) such that \( L(h) \varphi_1 L(f) \neq \varphi_2 L(f) \). Further, \( L(h) \varphi_1 L(f) = L(x_2) \) for \( x_1 : m \to m', i = 1, 2 \) and \( U(x_1) \neq U(x_2) \). Thus there is \( x \in Um \) such that \( V \overline{L}(x_1)(x) \neq V \overline{L}(x_2)(x) \), which implies that \( W'(\varphi_1) \neq W'(\varphi_2) \).

If we show that \( \overline{L} \) is \( W \)-covered, then the Theorem will follow from 3.4 applied to the concrete category \((A', W')\) instead of to \((A, V)\). Consider \( c \in C \) and \( x \in W' \overline{L}c \). There is \( m \in M \), \( \varphi : m \to c \) and \( x \in VTM \) with \( x = V \overline{L}(\varphi)(x) \) because \( \overline{L} \) is \( V \)-covered. Take \( m \) and \( h_0 \) for this \( \varphi \). Since \( \varphi = V \overline{L}(h_0)(x) \in Um \). Further \( \psi \in V \overline{L}(h_0 \varphi)(Um) \) because otherwise taking \( \nu \) and \( \nu' \) for this \( \psi \) we get a contradiction. Namely, \( \psi = V \overline{L}(\nu')(\psi) = V \overline{L}(\nu' h_0 \varphi)(x) = V \overline{L}(\nu' h_0 \varphi)(x) = V \overline{L}(\nu h_0 \varphi)(x) = V \overline{L}(\nu h_0 \varphi)(x) = V \overline{L}(\nu h_0 \varphi)(x) = V \overline{L}(\nu h_0 \varphi)(x) = \psi \). Hence \( \psi = V \overline{L}(h_0 \varphi)(\psi) \) for some \( \psi \in Um \). Suppose that \( V \overline{L}(h_0)(x) \neq V \overline{L}(h_0 \varphi)(x) \). For an \( h : c \to m \) and \( m' \in M \). By b) there are \( t : m \to m' \), \( t' : m \to m \) with \( V \overline{L}(t' h_0 \varphi)(\psi) = V \overline{L}(t' h_0 \varphi)(\psi) = V \overline{L}(t' h_0 \varphi)(\psi) = V \overline{L}(t' h_0 \varphi)(\psi) = V \overline{L}(t' h_0 \varphi)(\psi) = \psi \), which is a contradiction. Hence \( x = V \overline{L}(\varphi)(\psi) \) by 2.12 and the proof is accomplished.
3.6. Remarks: 1) Conditions ensuring in 3.4 and 3.5 that \( T \) exists and is \( V \)-covered, are yielded by 2.11 or 2.5 and 2.10.

2) If \( \{ U(f) \cup f: m \rightarrow c, m \in M \} \) is not always jointly epi, then \( U'c = \cup \{ U(f)(Um) \cup f: m \rightarrow c, m \in M \} \) defines a new concrete category \((C, U')\) which shares with \((C, U)\) all other suppositions of 3.4 or 3.5.

3) Condition b) can be replaced by the following one:

\[
\forall (\forall_0)(x_1, x_2) = \forall (\forall_0)(x_2) \text{ for } x_1, x_2 \in \mathcal{W}c \implies \forall (\forall_0)(x_2) = \forall (\forall_0)(x_1) \text{ for any } \forall: c \rightarrow m \text{ and } m \in M.
\]

3.7. Theorem. Let \((C, U)\) have the property of unicity, \((A, V)\) of transfer, \(M\) inductively and projectively generate \(C\), cogenerate \(C\), let \(A\) have and \(V\) compress colimits from \(M_\#\). Let \(M\) contain a \(1\)-free object \(e\) of \(C\) and for any \(m \in M, c \in C - M\) and \(\varphi: m \rightarrow c\) of \(C\) there exist a cogenerator \(n \in M\) of \(C\) and \(\forall_0: c \rightarrow n\) such that

\(a_1\) for any permutation \(t'\) of \(Um\) interchanging precisely two elements of \(Um\) there is a morphism \(t: n \rightarrow n\) of \(C\) such that \(U(t) = t'\),

\(a_2\) \(\text{card} (Um - U(\forall_0 \varphi)(Um)) = 1\)

and \(b)\) from 3.5 hold.

Then \(L_\#\) is a pseudorealization.

This theorem is a slight generalization of Theorem 1 from [10] (use 2.10 and 2.4). Conditions \(a_1\) and \(a_2\) imply \(a)\) (see [10]).
§ 4. Binding categories

If $\kappa$ is a regular cardinal, then a category $J$ is called $\kappa$-filtered when $J$ is not empty, to any family $\{ j_i \}$ of less than $\kappa$ objects of $M$ there is $j \in J$ and morphisms $j_i \to j$ for any $i$ and for any family $\{ f_i \}$ of less than $\kappa$ parallel morphisms of $J$ there is a morphism $f$ of $J$ equalizing all $f_i$. It follows that any diagram in $J$ having less than $\kappa$ morphisms is a base of a cone in $J$.

4.1. Lemma. Let $\mathcal{M}' = \{ m \in \mathcal{M} \}$ there is $c \in C - M$ and a morphism $m \to c$ in $C$ if be non-empty and contain any colimit in $C$ of a diagram in $\mathcal{M}'$ having less than $\kappa$ morphisms. Then the comma category $(X \downarrow c)$ is $\kappa$-filtered for any $c \in C$.

Proof is evident.

Let $\kappa$ be a regular infinite cardinal, $\kappa$ the complete graph (with loops) having $\kappa$ vertices and $1$, $2$ or $4$ the complete graph without loops having one, two or four vertices respectively. Let $\mathcal{C}_\kappa$ be the full subcategory of the category of undirected graphs composed of all connected 3-colourable graphs and the graph $m_\kappa$ having $\kappa$ and $4$ as components and $M_\kappa$ the full subcategory of $\mathcal{C}_\kappa$ determined by $1$, $m_\kappa$ and the graph $\nu_\kappa$ having $\kappa$ copies of $2$ as components. $\mathcal{C}_\kappa$ is binding for the category of connected 3-colourable graphs (see[11]). Let $\mathcal{U}$ be the usual forgetful functor of the category of graphs. Any restriction of $\mathcal{U}$ onto a full subcategory will be denoted by the same letter.
4.2. Theorem. Let \( (A, V) \) have the property of transfer, \( A \) have and \( V \) preserve finite limits and arbitrary intersections. Let \( \kappa \) be a regular infinite cardinal, let \( A \) have and \( V \) compress coequalizers and small \( \kappa \)-filtered colimits.

Then \( A \) is binding if and only if \( M_\kappa \) can be fully embedded into it.

Proof: If \( A \) is binding, then any small category can be fully embedded into it (see [3]). Let \( M_\kappa \) be fully embedded into \( A \). \( \{U(h)/h; c \to t\} \) is jointly monic for any \( c \in C_\kappa \) (see [11], (a) of Lemma 1). Thus, there is a pseudorealization \( T: M_\kappa \to A \) by Theorem 1.8 from [8] because \( 1, m_\kappa \in M_\kappa \) and \( t \) satisfies \( \alpha_1 \) from 3.6. \( M_\kappa \) projectively generates \( C_\kappa \), for \( m_\kappa \) does (see [11], (b) of Lemma 1) and inductively generates \( C_\kappa \), for \( \kappa \) does. Since there is no morphism from \( t \) to a 3-colourable graph, \( (K \downarrow c) \) is \( \kappa \)-filtered essentially by 4.1, which can be verified by consideration of coproducts of less than \( \kappa \) objects and coequalizers, because \( M \) need not contain coproducts of \( 1 \), for those are subgraphs of \( m_\kappa \) and any morphism from such a subgraph into a connected graph with at least two vertices can be extended to the whole \( m_\kappa \). Following 2.11, \( L \) exists and is \( V \)-covered. Finally, let \( m \in M_\kappa, c \in C_\kappa \) and \( g: m \to c \) be a morphism of \( C_\kappa \). Take \( m = m_\kappa \) and \( h_0 \): \( c \to m_\kappa \) such that \( \cup(h_0)(Uc) \subseteq \kappa \) and \( \cup(h_0)/\cup(q)(Um) \) is injective. Then a) holds because \( \alpha_1 \) and \( \alpha_2 \) from 3.7 are satisfied. b) holds for \( \{U(t)/t; m \to \kappa \} \) is jointly monic for any \( m \in M \) and \( h_0 \) can be factorized through \( h_0g \) for
any \( \mathcal{A} : c \rightarrow \mathcal{A} \). Therefore Theorem 4.2 follows from 3.5.

4.3. Corollary. Let \( \mathcal{A} \) be a regular infinite cardinal. An equational class \( \mathcal{A} \) of algebras having less than \( \mathcal{A} \)-ary operations is binding if and only if \( M_{\mathcal{A}} \) can be fully embedded into it.

4.4. Remark: 4.2 and 4.3 remain true if we suppose that \( M_{\mathcal{A}} \) can be pseudorealized into \( \mathcal{A} \) instead of \( M_{\mathcal{A}} \) fully embedded.

Let \( C \) be the full subcategory of the category of undirected graphs determined by all connected 3-colourable graphs and the graph \( \mathcal{A} \) and \( M \) the full subcategory of \( C \) composed of \( \mathcal{A} \) and \( \mathcal{A} \). \( M \) is the testing category from [11]. The following result is given in [10] for co-well-powered \( A \) and this additional supposition can be left out by 3.7. Similarly, co-well-poweredness can be omitted in Theorem 3 of [10].

4.5. Theorem. Let \( (\mathcal{A}, \mathcal{V}) \) have the property of transfer, let \( \mathcal{A} \) have and \( \mathcal{V} \) compress colimits from \( \mathcal{C}_{\mathcal{A}} \). Then \( \mathcal{A} \) is binding if and only if \( M \) can be fully embedded into it.

Finally, supposing the existence of many measurable cardinals we shall give an example of a non-binding monadic category containing any small category as a full subcategory. Let \( P_{\mathcal{V}} : En_{\mathcal{A}} \rightarrow En_{\mathcal{A}} \) be the contravariant power set functor, i.e. \( P_{\mathcal{V}} x = \exp x \) and \( P_{\mathcal{V}}(f)(x) = f^{-1}x \) for \( f : x \rightarrow y \) and \( x \leq y \).

4.6. Lemma. Let \( (\mathcal{A}, \mathcal{V}) \) be a monadic category and \( \mathcal{V} \) have a right adjoint. Then \( (\mathcal{A}^{op}, P_{\mathcal{V}}^{op}) \) is monadic.
Proof: Following [5], $(\operatorname{Emb}^\text{op}, P^\text{op})$ is monadic. Hence $P^\text{op}$ has a left adjoint and creates split coequalizers. Further, $V^\text{op}$ creates all colimits because $V$ is monadic. Putting all these facts together, $P^\text{op} V^\text{op}$ has a left adjoint and creates split coequalizers and therefore is monadic by Beck's precise tripleability theorem.

We recall that $(\mathcal{M})$ denotes the following assumption: There is a cardinal $\kappa$ such that every ultrafilter closed under intersections of $\kappa$ elements is trivial.

4.7. Example: Let non $(\mathcal{M})$ hold, $(\mathcal{A}, V)$ be the category of all algebras with two unary operations, where $V$ is the usual forgetful functor. By 4.6 $(\mathcal{A}^\text{op}, P^\text{op} V^\text{op})$ is monadic. Since $\mathcal{A}$ is binding (see [1]), any small category can be fully embedded into $A^\text{op}$. By [4] $\operatorname{Emb}^\text{op}$ cannot be fully embedded into $A$ and therefore $A^\text{op}$ is not binding.

By 4.5 and 4.7 there is no colimit compressing faithful functor $\operatorname{Emb}^\text{op} \rightarrow \operatorname{Emb}$, under non $(\mathcal{M})$.

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