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Each concrete category has a representation by $T_2$ paracompact topological spaces

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Abstract: It is shown that every concrete category can be fully embedded into a category whose objects are paracompact Hausdorff spaces and whose morphisms are all non-constant continuous (or closed continuous) mappings between these spaces.

Key words: Concrete category, full embedding, paracompact Hausdorff space, continuous mapping, closed continuous mapping.

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The aim of the paper is to prove that each concrete category is isomorphic to a category whose objects are paracompact connected Hausdorff spaces and whose morphisms are all non-constant continuous (closed continuous, respectively) mappings between these objects. The theorem is based on the fact that each concrete category is fully embeddable into $S(P_2)$ proved in [3] by Kučera.

A similar result was obtained by V. Trnková [5] who proved an analogical theorem for metric (or compact Hausdorff) spaces under the assumption of the non-existence proper class of measurable cardinals. The present results do not require any special set-theoretical assumption.
The author would like to express his gratitude to V. Trnková who introduced him to this problematics.

**Convention:** Denote $F_* = \langle -, A \rangle$ the contravariant hom-functor from the category of all sets and their mappings into itself.

**Definition.** Let $F$ be a contravariant functor from sets to sets. Denote $S(F)$ the category, objects of which are couples $(X, \mathcal{U})$, $X$ being a set, $\mathcal{U} \subseteq PX$, and $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ is a morphism if $f: X \rightarrow Y$ is a mapping with $Ff(\mathcal{V}) \subseteq \mathcal{U}$. In particular, objects of $S(P_2)$ are couples $(X, \mathcal{U})$, $\mathcal{U} \subseteq \exp X$ and morphisms $f: (X, \mathcal{U}) \rightarrow (Y, \mathcal{V})$ are mappings such that $f(A) \in \mathcal{U}$ for each $A \in \mathcal{V}$.

**Theorem 1.** Every concrete category can be fully embedded into the category $S(P_2)$.

Proof: see [3].

**Theorem 2.** There exists a metric continuum $M$ such that if $Z$ is a subcontinuum of $M$, $f: Z \rightarrow M$ is a continuous mapping then either $f$ is constant or $f(x) = x$ for all $x \in Z$. $M$ has pairwise disjoint subcontinua.

Proof: see [1].

**Convention:** For a given topological space $T$, $T^X$ denote, the topological product of topological spaces $T_i$, $i \in X$, where each $T_i$ is homeomorphic to $T$. Let $T_i$, $i \in I$ be topological spaces, then $\bigvee_{i \in I} T_i$ denote,
the topological sum of topological spaces \( T_i, i \in I \).

**Convention:** Denote \( \mathbb{Z} \) the set of all integers.

Choose arbitrary but fixed disjoint subcontinua \( A, B, C, z \in \mathbb{Z} \) of \( M \). Notice that the only continuous mappings between these three spaces are constants and the identities of \( A, B, C, z \in \mathbb{Z} \).

**Theorem 3.** There exists a full embedding \( \Phi : S(P_z) \rightarrow S(P_A) \).

**Proof:** see [4].

**Definition.** A topological space \( T \) is stiff if every continuous mapping \( f : T \rightarrow T \) is either the identity or a constant.

**Theorem 4.** Let \( T \) be a stiff Hausdorff space. Let \( f : T^A \rightarrow T \) be a continuous mapping. Then \( f \) is either a projection or a constant.

**Proof:** see [2].

**Corollary 5:** Let \( T \) be a stiff Hausdorff space. Then \( f : T^A \rightarrow T^R \) is a continuous mapping if and only if there exists a partial mapping \( \phi : R \rightarrow \mathbb{Q} \) and a point \( \alpha \in T^R, a = \{ \alpha_i \}_{i \in \mathbb{R}} \), such that for every \( x \in T^A \),

\[
f(x) = \psi = \{ \psi_i \}_{i \in \mathbb{R}} \quad \text{where} \quad \psi_i = x \phi(i) \quad \text{if} \quad \phi(i) \quad \text{is defined}, \quad \psi_i = \alpha_i \quad \text{otherwise}.
\]

In particular, \( f : T \rightarrow T^N \) is a continuous mapping if and only if there exists \( N' \subset N \) and \( \alpha = \{ \alpha_i \}_{i \in N} \in T^N \) such that \( f(x) = \psi = \{ \psi_i \}_{i \in N} \) and \( \psi_i = x \) if \( i \in N' \), \( \psi_i = \alpha_i \) otherwise.

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Corollary 6: The only continuous mappings between $A^N$ and either $B$ or $C_z$, $z \in Z$, are constants.

Lemma 7. Let $K$ be a subcontinuum of a Hausdorff space $Q$, let $a, b \in K$, $a \neq b$ such that $M = K - \{a, b\}$ is open in $Q$. Then for each continuous mapping $f : Z \to Q$, where $Z$ is a continuum, either there exists a component $H$ of $f^{-1}(K)$ such that $a, b \in f(H)$ or there exists a continuous mapping $\mathcal{F} : Z \to Q$ such that $\mathcal{F} = f$ on $f^{-1}(Q - M)$ and $\mathcal{F}(f^{-1}(K)) \subseteq \{a, b\}$.

Proof: see [5].

Construction 8: In each $C_z$, $z \in Z$, choose a pair distinct points $c_z, d_z$. Define a topological space $D = \bigvee_{z \in Z} C_z / \sim$, where $d_z \sim c_{z+1}$ for every $z \in Z$.

Choose distinct points $a_1, a_2 \in A$, $b_1, b_2 \in B$. For given set $X$ define a topological space $E_X = A^X \cup (B \times \langle 0, 1 \rangle) / \sim$, where $\langle 0, 1 \rangle$ is a discrete topological space and $a' = \{a'_x\}_{x \in X} \sim \{b_1, 0\} \cup \langle b_2, 0 \rangle \cup \langle b_2, 13 \rangle \cup \langle b_2, 1 \rangle \sim \overline{a} = \{\overline{a}_x\}_{x \in X}$.

For each object $P = (X, U)$ of $S(P_A)$ denote by $P^*$ the space $E_X \cup (D \times U)$, where $U$ is the discrete topological space with underlying set $U$. Let $\mathfrak{P}$ be a coarser topological space than $P^*$: a set $V$, open in $P^*$ is open in $\mathfrak{P}$ if and only if for each $\mu \in U \subset A^X$ either $\mu \notin V$ or there exists $n_0$ with $\bigcup_{n > n_0} C_n \times \{\mu\} \subset V$ and either $\{b_2, 0\} \notin V$ or there exists $n_1$ with $\bigcup_{n < n_1} C_n \times U \subset V$.

Clearly $\mathfrak{P}$ is a connected paracompact Hausdorff space.

Define a contravariant functor $\psi$ from $S(P_A)$ into the
category PAR of connected paracompact Hausdorff spaces: 
\[ \varphi P = \widetilde{P}, \quad \varphi f = (P_A \vee (A_b \times \{0,1\})) \vee (P_d \times P_A f / \mathcal{U}) / \sim, \]
where \(A_b\) and \(A_p\) are the identities of \(B\) and \(D\). 
Clearly, \(\varphi f\) is correctly defined and it is a closed continuous mapping. 
Evidently the functor \(\varphi\) is faithful.

Lemma 9. Let \(f: T \rightarrow \widetilde{P}\) be a non-constant continuous mapping.

a) If \(T = A\) then \(f(T) \subseteq A^X\);
b) if \(T = B\) then \(f(T) = B \times \{i\}\), where \(i \in \{0,1\}\).
c) If \(T = C_2\) then \(f(T) = D \times \{\mu\}\) for some \(\mu \in \mathcal{U}\).

In all above cases, \(f\) is an embedding.

Proof: Let \(X, \alpha, \beta\) denote one of the following:

a) \(X = C_{z_1} \times \{\mu\}, \alpha = (c_z, \mu), \beta = (d_z, \mu)\) for some \(z \in Z, \mu \in \mathcal{U}\).

b) \(X = B \times \{i\}, \alpha = (b_{i}, i), \beta = (b_{i}, i)\) for some \(i \in \{0,1\}\).

Suppose that the former case in Lemma 7 takes place, i.e. that there is a component \(L\) of \(f^{-1}(X)\) with \(\alpha, \beta \in f(L)\). Then we get easily by Theorem 2 that \(L\) is homeomorphic to \(T\) and \(f\) is a homeomorphism of \(T\) onto \(X\). Now, suppose that, for all \(X, \alpha, \beta\) as above, the latter case in Lemma 7 takes place.

1) Suppose that \(f(T)\) meets the interior of some \(X\), where \(X\) is from a). Then apply Lemma 7 on \(f, X' = C_{z-1} \times \{\mu\}, \langle c_{z-1}, \mu \rangle, \langle d_{z-1}, \mu \rangle\) to obtain \(X\)
and again Lemma 7 to $F, X' = C_{x+1} \times \{\omega\}, <c_{x+1}, \omega>, <d_{x+1}, \omega>$ to obtain $F$. Then $F$ coincides with $f$ on $f^{-1}(X)$ and $F(T)$ is a continuum which does not meet the interiors of both $X'$ and $X''$ but it meets the interior of $X$. Then, as easily seen from the construction of $F$, $F(T) \subset X$.

By Theorem 2, $F$ is an embedding of $T$ onto $X$ and $f = F$.

2) Let the assumption of 1) not hold. Then $F(T) \subset A^X \cup B \times \{0, 1\}$ as for any continuum which does not meet the interior of any $X$ from a).

Let us apply Lemma 7 on $f, B \times \{0\}, <E_1, 0>, <E_2, 0>$ to obtain $F$ and again Lemma 7 on $F, B \times \{1\}, <E_1, 1>, <E_2, 1>$ to obtain $F'$.

If $F$ is constant then clearly $F(T) \subset B \times \{0\}$ and $f$ is an embedding by Theorem 2. Analogously, if $F'$ is constant then $F'$ is an embedding of $T$ onto $B \times \{1\}$ and so is $f$. Let $F$ be non-constant. As $F(T) \subset A^X$, we may apply Corollaries 5, 6. We obtain that $F$ is an embedding of $T$ into $A^X$ and so is $f$.

**Lemma 10.** Let $f : P \longrightarrow \mathbb{R}$ be a continuous mapping $P, R \in S(P_A)$ with $f/B \times \{0\} = B \times \{0\}$. Then there exists $\varphi : R \longrightarrow P$ such that $\varphi \varphi = f$.

**Proof:** Lemma 9 implies either $f/B \times \{1\} = B \times \{1\}$ or $f(B \times \{1\} = <E_1, 1>$. If $f(B \times \{1\} = <E_1, 1>$ then $f(\mathbb{R}) = <E_1, 1>$ and therefore there exists $\alpha : A \longrightarrow \mathbb{R}$ such that $<E_1, 0>, <E_2, 0> \in \alpha(A)$ but this is impossible.

Hence $f/B \times \{1\} = A \times \{1\}$. Denote $\Delta_X$ the diagonal of $A^X$, $\Delta_Y$ the diagonal of $A^Y$, where $P = (X, \mathcal{U})$, 

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R = (Y, \mathcal{V})$. We have $f(\Delta_X) = \Delta_Y$ and so $f(A^X) \subseteq A^Y$.

Corollary 5 implies that there exists $\varphi : Y \to X$ such that $f/A^X = \mathcal{P}_A \varphi$. As $f(\langle \mathbb{E}_1, 1 \rangle) = \langle \mathbb{E}_1, 1 \rangle$ and $f(A^X) \subseteq A^Y$, $f/D \times \{ \omega \}$ is an embedding from $D \times \{ \omega \}$ into $D \times \{ f(\omega) \}$ and therefore $f/D \times \mathcal{U} = \mathcal{P}_D \mathcal{P}_A \varphi/\mathcal{U}$, and $\mathcal{P}_A \varphi(\mathcal{U}) \subseteq \mathcal{V}$. Hence $\psi \varphi = f$.

**Lemma 11.** Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a continuous mapping such that $f/B \times \{ 0 \} \neq f_B \times \{ 0 \}$. Then $f$ is constant.

**Proof:** Assume that $f/B \times \{ 0 \}$ is non-constant. Then Lemma 9 implies that $f/B \times \{ 0 \}$ is an embedding and so $f(\langle x, 0 \rangle) = \langle x, 1 \rangle$ for every $x \in B$. Therefore $f(\langle \mathbb{E}_1, 1 \rangle) = f(\langle \mathbb{E}_2, 0 \rangle) = \langle \mathbb{E}_2, 1 \rangle$ and by Lemma 9 we have $f(B \times \{ 1 \}) = \langle \mathbb{E}_2, 1 \rangle$. Hence $\langle \mathbb{E}_2, 1 \rangle \in f(\Delta_X)$ and $\langle \mathbb{E}_2, 0 \rangle \notin f(\Delta_X)$ which is a contradiction (see Lemma 9). Therefore $f/B \times \{ 0 \}$ is constant by Lemma 9. Analogously $f/A^X$ is constant and so is $f/\Delta_X$. Therefore $f/A^X$ is constant by Lemma 9 and so is $f$.

**Definition.** Let $\mathcal{X}, \mathcal{L}$ be concrete categories. A functor $\mathcal{D} : \mathcal{X} \to \mathcal{L}$ is an almost full embedding of $\mathcal{X}$ into $\mathcal{L}$ if $\mathcal{D}$ is an embedding of $\mathcal{X}$ onto a subcategory of $\mathcal{L}$ whose objects are $\mathcal{D}(\alpha)$, $\alpha$ running over objects of $\mathcal{X}$ and whose morphisms are all non-constant $\mathcal{L}$-morphisms between these objects.

**Theorem 12.** Denote $\text{PAR}$ the category of paracompact connected Hausdorff spaces and continuous mappings, $\text{PAR}_c$ its subcategory with the same objects and continuous closed mappings as morphisms. Then each category $\mathcal{L}$...
is almost universal in the sense that each concrete category has an almost full embedding into $L$.

Theorem 12 follows from Construction 9 and Lemmas 10 and 11.

A class $C$ of topological spaces is called stiff for every continuous mapping $f : T \rightarrow T'$, with $T, T' \in C$, is either constant or the identity of the space $T = T'$ onto itself.

V. Trnková had constructed a stiff class (not a set) of paracompact spaces as follows.

Let $H_i$, $i = 1, \ldots, 5$ be five disjoint subcontinua of the Cook continuum. Choose points $a, b, c_2, c_3 \in H_1$, $c_1, c_4 \in H_2, i = 2, \ldots, 5$, all distinct. For each ordinal $\alpha$ and $i = 1, \ldots, 5$, put $H_i^\alpha = \{ (x, \alpha) | x \in H_i \}$,

$\mathcal{G}_i^\alpha(x, \alpha) = x$. We write $x^\alpha$ instead of $(x, \alpha)$. Let $\omega$ be an ordinal. Put

$$Q_\omega = (\bigcup_{\alpha} H_1^\alpha \setminus \{ x_1^\alpha \}) \cup (\bigcup_{i=2,3} H_i^\alpha \setminus \{ x_i^\alpha, n_i^\alpha \}) \cup (x_4^0 \setminus \{ x_4^0, n_4^0 \}) \cup H_5^\omega).$$

$G \subset Q_\omega$ is open iff it fulfills (1) - (5).

(1) $G_i^\alpha (G \cap H_i^\alpha)$ is open in $H_i^\alpha$ for all $i = 1, \ldots, 5$,

$\alpha \leq \omega$;

(2) if $\alpha \in \omega, \alpha' \in G$ then

$G_i^\alpha (G \cap H_i^\alpha)$ is a nbh of $x_i$ in $H_i^\alpha$ whenever $\alpha = 0$.
\( \varphi_\alpha (G \cap H_\alpha) \) is a nbh of \( K_\beta \) in \( H_\beta \) whenever
\[ \alpha = \beta + 1 \]
\( G \) contains \( H_\gamma \) for all \( \alpha' \leq \gamma \leq \alpha \) (and some \( \alpha' < \alpha \)) whenever \( \alpha \) is limit;

(3) if \( \alpha \in \omega \), \( i = 2, 3 \), \( \kappa_\alpha \in G \), then \( \varphi_\alpha (G \cap H_\alpha) \) contains a nbh of \( \kappa_\alpha \) in \( H_\beta \);

(4) if \( \kappa_5 \in G \), then \( G \) contains \( H_\gamma \) for all \( \alpha' \leq \gamma < \omega \) (and some \( \alpha' < \omega \)).

(5) if \( \kappa_\omega \in G \), then \( \varphi_\alpha (G \cap H_\alpha) \) contains a nbh of \( \kappa_\omega \) in \( H_\alpha \) for all \( (i, \alpha) = (0, 4), (\omega, 5) \) or \( i = 2, 3 \) and \( \alpha \in \omega \).

By means of Lemma 7, one can prove that \( \{ G_\alpha | 1 \leq \alpha \} \) is a stiff proper class of paracompact spaces.

The existence of a stiff proper class of paracompact spaces follows also from the main result because "large discrete category" can be almost fully embedded in \( \text{PAR} \).

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