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NOTE ABOUT ATOM-CATEGORIES OF TOPOLOGICAL SPACES

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Abstract: Minimal members of the "lattice" of epireflective subcategories of topological spaces are investigated. They are in a close connection with subspaces of Čech-Stone compactifications of discrete spaces.

Key-words: Epireflective subcategory, Čech-Stone compactification.

AMS: 18A40, 54C20

All topological spaces are assumed to be completely regular Hausdorff; the category of all such spaces together with continuous mappings will be denoted by $\text{Top}_{\text{CR}}$.

We are going to investigate the ordering given by inclusion between epireflective subcategories of $\text{Top}_{\text{CR}}$ (by Kennison theorem, $[K]$, between closed-hereditary and productive classes of topological spaces). We shall use without references simple facts about epireflective subcategories (see e.g.: $[M_2],[H_4],[H_2]$). The epireflective categories $\mathcal{K}(E)$ of $E$-compact spaces will play a great role in the sequel ($E$-compact spaces, $[M_4]$, are homeomorphs of closed subspaces of powers $E^\omega$). The first fact that is relevant to our consideration is due to Mrówka, $[M_2]$: Let $\mathcal{N}$ be a countable discrete topological space and let $D(2)$ be a two-point discrete space; then there is no
epireflective subcategory $\mathfrak{K}$ such that $\mathfrak{K}(D(2)) \not\subseteq \mathfrak{K} \not\subseteq \mathfrak{K}(N)$. We take this property as a foundation for the following definition:

**Definition.** Let $\mathfrak{K}$, $\mathfrak{L}$ be epireflective subcategories of $\text{Top}_{CR}$. Then $\mathfrak{L}$ is said to be an atom-category above $\mathfrak{K}$ if $\mathfrak{L} \nsubseteq \mathfrak{K}$ and there is no epireflective subcategory $\mathfrak{M}$ of $\text{Top}_{CR}$ such that $\mathfrak{K} \nsubseteq \mathfrak{M} \subseteq \mathfrak{L}$.

Atom-categories above $\mathfrak{K}(D(2))$ will be called briefly atom-categories. The Mrówka's result quoted above asserts that $\mathfrak{K}(N)$ is an atom-category. It is clear that atom-categories are of the form $\mathfrak{K}(E)$ for a suitable space $E$ and that they are minimal in the sense that the only epireflective subcategories of $\text{Top}_{CR}$ strictly contained in them are the categories $\mathfrak{K}(D(2))$ and $\mathfrak{K}(D(4))$. R. Blelko was interested in the question whether $\mathfrak{K}(T\omega_\xi)$ are atom-categories ($T\omega_\xi$ is the ordered space of all ordinals less than $\omega_\xi$); the answer was negative $[B_1],[B_2]$, if $\omega_\xi \neq \omega_0$, of course. Nevertheless, it is proved in [P] that there is an atom-category $\mathfrak{K}(A_\xi)$ contained in $\mathfrak{K}(T\omega_\xi)$ for any $\omega_\xi$ and, moreover, $A_\xi$ can be chosen in such a way that $\mathfrak{K}(P \times A_\xi) (\text{card } P > 2)$ is an atom-category above $\mathfrak{K}(P)$ for regular ordinals $\omega_\xi$ provided $\text{comp } \mathfrak{K}(P) > \omega_\xi$ (by $\text{comp } \mathfrak{B}$, $\mathfrak{B}$ a class of topological spaces, we mean $\text{min } \{ \alpha \mid \exists X \in \mathfrak{B}, \exists A \subseteq X, \text{card } A = \alpha, \overline{X} \text{ is not compact} \}$ if it exists, i.e., if $\mathfrak{B}$ contains non-
compact spaces).

The aim of this paper is to exhibit other examples of atom-categories and to give properties of a topological space $\mathcal{E}$ sufficient for $\mathcal{X}(\mathcal{E})$ to contain an atom-category.

We have mentioned that atom-categories are simple but we can say more about "generators" of such categories

($\beta P$ is the Čech-Stone compactification of $P$, $\mathcal{X} = \bigcup \{A \mid A \subset X, \text{card } A < \alpha \}$;)

Proposition 1: Let $\mathcal{X}$ be an atom-category containing noncompact spaces. Then there is an object $X$ of $\mathcal{X}$ such that $\mathcal{X}(X) = \mathcal{X}, D \subset X \subseteq \beta D, \mathcal{X}^{\beta D} = X$ where $D$ is a discrete space of cardinality $\alpha = \text{comp } \mathcal{X}$.

Proof: Put $X = \beta_\mathcal{X} D$, the reflection of $D$ in $\mathcal{X}$.

We do not know whether the following converse of Proposition 1 is true: Let $D$ be a discrete space of cardinality $\alpha$, $D \subset X \subseteq \beta D, \mathcal{X}^{\beta D} = X$, $\beta_\mathcal{X}(X)D = X$, then $\mathcal{X}(X)$ is an atom-category.

We can prove the converse in special cases, e.g. if $\alpha = \omega_0$ or $X = \mathcal{X}^{\beta D}$, $\text{card } D = \alpha$ is regular.

($P$ is a strongly discrete subset of $\mathcal{X}$ if there is a disjoint open family $\{U_p \mid p \in P\}$ in $\mathcal{X}$ with $p \in e \subseteq U_p$.)

Theorem 1. Suppose that $D$ is a discrete space of cardinality $\alpha$, $D \subset X \subseteq \beta D, \mathcal{X}^{\beta D} = X$, $\beta_\mathcal{X}(X)D = X$ and
that each subset of $X$ of cardinality $\omega$ and with non-compact closure in $X$ contains a strongly discrete subset of the same cardinality. Then $\mathcal{K}(X)$ is an atom-category.

**Proof:** Let $E \in \mathcal{K}(X)$, $E$ be noncompact ($\mathcal{K}(D(2))$ is a class of all compact spaces contained in $\mathcal{K}(X)$). We have to prove that $X \in \mathcal{K}(E)$. We may suppose that $E$ is a closed subspace of $X$. There is an $i \in I$ such that $\nu_i[E]^X$ is not compact and, thus, $\text{card} \; \nu_i[E] \geq \omega$. By the assumption, there is a strongly discrete subset $A$ of $\nu_i[E]$, $\text{card} \; A = \omega$, with the corresponding disjoint open family $\{U_{\omega}\}$. Making use of the equality $\beta_{\mathcal{K}(X)} D = X$ we can prove that $X$ is homeomorphic to $X$ (if $\varphi : A \longrightarrow D$ is bijective, there is an $\varepsilon : D \longrightarrow D$ such that the continuous extension $\varepsilon$ on $\beta D$ into $\beta D$ extends $\varphi$; then $\varepsilon / X$ is the homeomorphism). Now, let $\varphi : A \longrightarrow E$ be a bijective mapping with the inverse $\nu_{\varphi}[A]$. There exists a continuous extension $\varphi : X \longrightarrow E$ which must be a homeomorphism then. Consequently, $X$ can be embedded as a closed subspace into $E$.

As mentioned above, the condition about strongly discrete subsets is clearly fulfilled if $\omega = \omega_0$ or if $X = D \beta$, $\text{card} \; D = \omega$ is regular. In the second case we receive atom-categories $\mathcal{K}(X)$ contained in $\mathcal{K}(T \omega_\alpha)$ and described in [F]. The first can give:

**Theorem 2.** If $\mathcal{K}$ is an epireflective subcategory of $\text{Top}_{CR}$ containing an object which is not strongly...
countably compact (i.e., \( \text{comp} \mathcal{K} = \omega_0 \)), then there exists an atom-category \( \mathcal{L} \subseteq \mathcal{K} \).

We do not know whether Theorem 2 holds generally without any assumption on \( \text{comp} \mathcal{K} \). To prove a more general version one must remove condition on strongly discrete subsets in Theorem 1 because as Hajnal and Juhász [HJ] proved under generalized continuum hypothesis, for any infinite cardinal \( \alpha \), there exists a set \( A \) in \( \beta D \), \( \text{card} D = \alpha \), such that \( \text{card} A = 2^{2^\alpha} \) and no uncountable \( B \subseteq A \) is strongly discrete.

Theorem 1 for \( \alpha = \omega_0 \) suggests the following construction of spaces \( X \) generating atom-categories (we write \( \mathcal{L} \) for the continuous extension of \( \mathcal{L} : N \rightarrow \beta N \) on \( \beta N \)). Let \( X_0 = N \) and all \( X_\xi \), \( \xi < \eta \), be defined; then we put \( \mathcal{L} \subseteq \bigcup_{\xi < \eta} X_\xi \).

It is easy to prove the following properties of \( \{ X_\xi \} \):
- if \( \xi \leq \eta \), then \( X_\xi \subseteq X_\eta \);
- if \( \mathcal{L} : N \rightarrow X_\xi \), then \( \mathcal{L} [X_\xi] \subseteq X_{\xi + 1} \);
- \( X_{\omega_1} = \bigcup_{\xi < \omega_1} X_\xi \). It follows that
  - \( X_{\omega_1 + 1} = X_{\omega_1} \)
  - \( \beta_{\mathcal{K}(X_{\omega_1})} N = X_{\omega_1} \), i.e. by Theorem 1 that \( \mathcal{K}(X_{\omega_1}) \) is an atom-category provided \( X_{\omega_1} \neq \beta N \). This last condition is guaranteed by the assumption \( \text{card} X_0 < 2^{\omega_0} \) (then \( \text{card} X_{\omega_1} \leq 2^{\omega_0} \)). One can deduce that there is exactly \( 2^{2^{\omega_0}} \) different atom-categories \( \mathcal{K}(X) \) generated by the spaces \( X \) with properties \( N \subseteq X \subseteq \beta N \), \( \beta_{\mathcal{K}(X)} N = X \), \( \text{card} X \leq 2^{\omega_0} \).
**Remark:** Proposition 10 of [P] can be generalized: Let \( \mathfrak{K}(A) \) be an atom-category, \( \text{comp} \, A = \alpha, D(\alpha) \subset A \subset \beta D(\alpha) \).

Suppose that each non-compact set of \( A \) contains a strongly discrete subset of cardinality \( \alpha \). Let \( \mathcal{P} \) be an epireflective subcategory of \( \text{Top}_{\text{CR}} \), \( \text{comp} \, \mathcal{P} > \alpha \). Denote by \( \mathfrak{K}(A) \vee \mathcal{P} \) the least epireflective subcategory of \( \text{Top}_{\text{CR}} \) containing both \( \mathfrak{K}(A) \) and \( \mathcal{P} \). This category is an atom-category above \( \mathcal{P} \).

**References:**


[H] HERRLICH H.: \( \epsilon \) -kompakte Räume, Math Z. 96 (1967), 228-255.


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