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CONTINUITY OF NĚMYSKIJ’S OPERATOR IN HÖLDERSpaces

Pavel Drábek, Praha

Abstract: This paper deals with the investigation of the mapping \( u(x) \mapsto f(u(x)) \), where \( f \) is a given real-valued function. There are proved the necessary and sufficient conditions upon \( f \) to be \( f(u(x)) \) Hölder-continuous function for an arbitrary Hölder-continuous function \( u \) and, moreover, the necessary and sufficient conditions for the mapping considered to be continuous between the spaces of Hölder-continuous functions.

Key-words: Spaces of Hölder-continuous functions, Němyskij’s operator.

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1. Introduction. Let \( M, N \) be positive integers. Denote by \( \mathbb{R}^M \) and \( \mathbb{R}^N \), respectively, the \( M \)-dimensional and \( N \)-dimensional, respectively, Euclidean spaces with the norms \( \| \cdot \|_M \) and \( \| \cdot \|_N \), respectively. Let \( \Omega \) be an open bounded non-empty subset of \( \mathbb{R}^N \). For \( \alpha \in (0,1) \) define \( H^\alpha_{\infty}(\Omega) \) the (so-called Hölder) space of all mappings \( u: \Omega \mapsto \mathbb{R}^M \) defined on \( \Omega \) and with values in \( \mathbb{R}^M \) such that

\[
\| u \|_{H^\alpha_{\infty}} = \sup_{x \in \Omega} \| u(x) \|_M + \sup_{x \neq y \in \Omega} \frac{\| u(x) - u(y) \|_M}{\| x - y \|_N^{1+\alpha}} < + \infty
\]

It is easy to see that \( H^\alpha_{\infty}(\Omega) \) is a Banach space with
the norm $\| \cdot \|_{H^M}$.

Instead of $H^M_\infty(\Omega)$ we shall write $H^M_\infty(\Omega)$ only.

Let $f: \mathbb{R}^M \rightarrow \mathbb{R}^1$ be a real valued function. For $u \in H^M_\infty(\Omega)$ denote by $\mathcal{N}(u)$ the function defined on $\Omega$ by the relation

$$\mathcal{N}(u)(x) = f(u(x)), \quad x \in \Omega.$$ 

The mapping $\mathcal{N}$ is usually called the Nemyckij's operator. In this paper, we give the necessary and sufficient conditions upon $f$ to be $\mathcal{N}(u) \in H^M_\infty(\Omega)$ for any $u \in H^M_\infty(\Omega)$ (see Theorem 1) and also the necessary and sufficient conditions upon $f$ to be the mapping $\mathcal{N}$ continuous from the space $H^M_\infty(\Omega)$ into $H^M_\infty(\Omega)$ (see Theorem 2). It is interesting that if $\mathcal{N}$ works from $H^M_\infty(\Omega)$ into $H^M_\infty(\Omega)$, then it is not generally continuous. This is a quite different result than for the space $C(\overline{\Omega})$ of continuous functions or for the space $L_p(\Omega)$ of $p$-integrable measurable functions (see e.g. [1],[2]).

2. **Necessary and sufficient conditions for**

$$\mathcal{N}(H^M_\infty(\Omega)) \subset H^M_\infty(\Omega).$$

Let $\mathbb{N}$ denote the set of all positive integers.

**Lemma 1.** Let $\alpha \in (0,1)$ and let $\{a_n^i\}^\infty_{n=1}$ be the real convergent sequences. Then for each $\varepsilon \in (0,1)$ there exists an increasing function $k$ defined on $\mathbb{N}$ with values in $\mathbb{N}$ and an increas-
ing sequence \( \{ t_n \} \in (0, \varepsilon) \) such that for each \( m, n \in \mathbb{N} \) it is
\[
\sum_{i=1}^{M} |a_k(n) - a_k(m)| \leq |t_n - t_m| \varepsilon.
\]

**Proof.** Let \( \varepsilon > 0 \). If \( n \in \mathbb{N} \) then there exists \( k(n) \) such that for each \( m > k(n) \) and \( i = 1, 2, \ldots, M \) it is
\[
\sum_{i=1}^{M} |a_k(n) - a_k(m)| < \varepsilon / 2^n.
\]
It is possible to construct \( k \) to be an increasing function. Put
\[
t_n = \sum_{i=1}^{2^n} \frac{\varepsilon}{2^n}.
\]
Now, one can immediately see that \( t_n \) is the wanted sequence.

**Lemma 2.** Let the operator \( \mathcal{N} \) map \( H^M_\infty(\Omega) \) into \( H^\infty(\Omega) \). Then \( f \) is a continuous function on \( \mathbb{R}^M \).

**Proof.** Let us suppose that \( f \) is not continuous at the point \( a_0 = [a_0^1, \ldots, a_0^M] \in \mathbb{R}^M \).
Then there exists \( \omega_0 > 0 \) and a sequence \( a_n = [a_n^1, \ldots, a_n^M] \in \mathbb{R}^M \) \((n = 1, 2, \ldots)\) such that \( \lim_{n \to \infty} \| a_n - a_0 \|_M = 0 \) and
\[
|f(a_n) - f(a_0)| \geq \omega_0
\]
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for all $n \in \mathbb{N}$.

Choose $x_0 = [x_0^1, \ldots, x_0^N] \in \Omega$ arbitrary but fixed and let $K_{2\varepsilon}(x_0) \subset \Omega$ where $K_{2\varepsilon}(x_0)$ is a ball centered at $x_0$ and with the radius $2\varepsilon$.

Let $\{t_n\}$ and $k$ have the same meaning as in Lemma 1. Put

$$x_n = x_0 + [t_n, 0, \ldots, 0] = [x_0^1 + t_n, x_0^2, \ldots, x_0^N].$$

Obviously $x_n \in K_{2\varepsilon}(x_0), (n \in \mathbb{N})$ and $\{x_n\}$ converges to some $z \in K_{2\varepsilon}(x_0)$. For each $n, m \in \mathbb{N}$ we have

$$|a^i_k(n) - a^i_k(m)| \leq |t_n - t_m|^\infty = \|x_n - x_m\|_\infty.$$ 

Let us define

$$u(x_n) = a^i_k(n), \quad n = 1, 2, \ldots; \quad u(z) = a_0,$$

and denote $\mathcal{M} = \{x_n\} \cup \{z\}$. According to (2) $u$ is bounded on the closed set $\mathcal{M}$ and satisfies the Hölder condition on $\mathcal{M}$. Thus (see e.g. [3, Proposition 1]) there exists a function $U$ defined on $\bar{\Omega}$ such that $U \in H^M(\Omega)$, the restriction of $U$ on $\mathcal{M}$ is $u$,

$$\sup_{x \in \mathcal{M}} \|U(x)\|_M = \sup_{x \in \mathcal{M}} \|u(x)\|_M,$$

and

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It means that $g$ is a continuous function on $\Omega$, especially,

$$f(a_0) = g(a_0) = \lim_{n \to \infty} g(x_n) = \lim_{n \to \infty} f(u(x_n)) = \lim_{n \to \infty} f(a_k(n)),$$

which is a contradiction with (1).

**Theorem 1.** The operator $\mathcal{N}$ maps $H^M(\Omega)$ into $H^M(\Omega)$ if and only if $f$ is a locally lipschitzian function from $\mathbb{R}^M$ into $\mathbb{R}$.

**Proof.** If $f$ is locally lipschitzian, then we immediately obtain $\mathcal{N}(H^M(\Omega)) \subset H^M(\Omega)$.

On the other hand, let us suppose that $\mathcal{N}$ maps $H^M(\Omega)$ into $H^M(\Omega)$ and that $f$ is not a locally lipschitzian function. Then there exist bounded sequences $\{\xi_n\}, \{\eta_n\} \subset \mathbb{R}^M$ such that

$$|f(\xi_n) - f(\eta_n)| > n \|\xi_n - \eta_n\|_M$$

for $n \in \mathbb{N}$. We can suppose that $\lim_{n \to \infty} \xi_n = \xi_0$ and $\lim_{n \to \infty} \eta_n = \eta_0$. It is $\xi_0 = \eta_0$, for Lemma 2 implies

$$|f(\xi_n) - f(\eta_n)| \leq \text{constant}.$$
and thus
\[ \text{constant} > n \| \xi_n - \eta_n \|_M \]
for each \( n \in \mathbb{N} \).

In order that the following construction be simpler, we can suppose (without loss of generality) that \( \Omega \) is an open ball in \( \mathbb{R}^N \) centered at the origin and the radius of which is \( r > 0 \).

Consider an open ball \( K_k \subset \mathbb{R}^M \), \( k \in \mathbb{N} \), \( \frac{\lambda}{2^{n+2}} \leq 1 \) centered in \( \xi_0 \) and with radius \( \frac{\lambda}{2^{n+2}} \). There exists \( n_1 \in \mathbb{N} \) so that \( \xi_{m_1} \) and \( \eta_{m_1} \) are situated in \( K_k \) (denote \( \xi_{m_1} = \xi_1^t \) and \( \eta_{m_1} = \eta_1^t \)). Denote \( A_1 = \{ [t_1, 0, \ldots, 0] \mid t \in \mathbb{R}^1 \} \) and \( x_1 = [0, 0, \ldots, 0] \in \mathbb{R}^N \). Let \( y_1 \in A_1 \), and \( y_1 = [t_1, 0, \ldots, 0] \), \( t_1 > 0 \) and, moreover,
\[ \| \xi_1^t - \eta_1^t \|_M = \| x_1 - y_1 \|_N. \]

It is obvious that
\[ \| x_1 - y_1 \|_N < \frac{\lambda}{2^{n+2}}. \]

Further put
\[ K_{k+1} = \{ z \in \mathbb{R}^M \mid \| z - \xi_0 \|_M < \frac{1}{2^{n+3}} \}. \]

There exist \( \xi_2^t \in \{ \xi_m^t \} \), \( \eta_2^t \in \{ \eta_m^t \} \) such that \( \xi_2^t \),
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Let $x_2 \in A_1$, $y_2 \in A_1$ be such that $x_2 = [\tau_2, 0, \ldots, 0] \in \mathbb{R}^N$, $t_2 > \tau_2$ and $\|x_2 - y_2\|_M = \|x_2 - y_2\|_N$.

Obviously, $\|x_2 - y_2\|_N < \frac{\mu}{2^{m+2}}$.

Consider an open ball $K_{k+m} = \{ z \in \mathbb{R}^M | \| z - f \|_M < \frac{\mu}{2^{m+2}} \}$. There exist $f_{m+1} \in \{ f_m \}$, $\eta_{m+1} \in \{ \eta_m \}$ such that $f_{m+1} \in K_{k+m}$, $\eta_{m+1} \in K_{k+m}$. Let $x_m \in A_1$, $y_m \in A_1$, be such that $x_m = [\tau_m, 0, \ldots, 0] \in \mathbb{R}^N$, $t_m > \tau_m$ and $\|x_m - y_m\|_M = \|x_m - y_m\|_N$.

Clearly $\|x_m - y_m\|_N < \frac{\mu}{2^{m+2}}$.

Let $x_0$ mean the limit of the sequence $\{x_n\}$, where $z_{2^L} = x_L$, $z_{2^L-1} = y_L$ ($L \in \mathbb{N}$). We have

$$\|x_0\|_N \leq 2 \sum_{i=1}^{N} \frac{\mu}{2^{i+1}} \leq 2 \cdot \frac{\mu}{2} \cdot \frac{1}{2^{L+1}} = r,$$

which means that the whole sequence $\{z_n\}$ and also $x_0$ are situated in $\Omega$. Let us define the function $u_1 = [u_1^1, u_1^2, \ldots, u_1^M]$ as follows:
\[ u_1(x_m) = \xi_{m}^1, \quad m = 1, 2, \ldots, \]
\[ u_1(y_m) = \eta_{m}^1, \quad m = 1, 2, \ldots, \]
\[ u_1(x_0) = \xi_0. \]

Put \( Z = \{ z_{n}^{\infty}_{n=1} \cup \{ x_0 \} \}. \) The set \( Z \subset \mathbb{R}^N \) is closed and \( u_1 \) satisfies on \( Z \) the Hölder condition with the exponent \( \alpha \). Indeed, we can suppose (without loss of generality) that \( m \leq n \) and thus:

a) \[ \| u_1(x_m) - u_1(x_n) \|_M = \| \xi_{m}^1 - \xi_{n}^1 \|_M < \frac{\kappa}{2^{k+n}} \]

and if \( \| x_m - x_n \|_N = 1 \) then \( \frac{\kappa}{2^{k+n}} \leq \| x_m - x_n \|_N \),

and if \( \| x_m - x_n \|_N < 1 \) then \( \| x_m - x_n \|_N \)

b) \[ \| u_1(x_m) - u_1(y_n) \|_M = \| \xi_{m}^1 - \eta_{n}^1 \|_M < \frac{\kappa}{2^{k+n}} \]

for \( m < n \);

for \( m = n \) we have \( \| \xi_{m}^1 - \eta_{m}^1 \|_M = \| x_m - y_m \|_N \);

c) \[ \| u_1(x_m) - u_1(x_0) \|_M < \frac{\kappa}{2^{k+m}} \leq \| x_m - x_0 \|_N \]

and further we use the same arguments as in a);
d) analogously as in c) we estimate

\[ \| u_1(y_m) - u_1(x_0) \|_M < \frac{\kappa}{2^{M+1}}. \]

By using [3, Proposition 1], there exist the extensions of \( u_i \), \( i = 1, \ldots, M \), so that \( u_i \in H^\infty(\Omega) \) for \( i = 1, \ldots, M \). It means that \( u = [u_1, \ldots, u^M] \in H^M(\Omega) \). But for each \( K > 0 \) there exists \( m \in \mathbb{N} \) such that

\[ |f(u(x_m)) - f(u(y_m))| = |f(x_m^i) - f(\eta_m^i)| \leq \]

\[ \geq K \| x_m^i - \eta_m^i \|_M = K \| x_m - y_m \|_M^\infty, \]

and so \( f \circ u \in H^\infty(\Omega) \). This contradiction completes the proof of Theorem 1.

3. Necessary and sufficient conditions for continuity of \( \eta \)

Lemma 3. Let the partial derivatives of the first order of the function \( f \) be continuous on \( \mathbb{R}^M \) and let \( \Omega \) be a bounded subset of \( \mathbb{R}^M \). Then for each \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that for \( a \in \Omega \) and \( h \in \mathbb{R}^M \) with \( 0 < \| h \|_M < \delta \) it is

\[ \left| \frac{f(a + h) - f(a)}{\| h \|_M} - \sum_{i=1}^M \frac{\partial f(a)}{\partial f_i} \cdot \frac{h_i}{\| h \|_M} \right| < \varepsilon. \]

Proof. The uniform continuity on \( \Omega \) of the partial
derivatives implies that for each $\varepsilon > 0$ there exists $\sigma^* > 0$ such that for each $a \in 0^\prime$, $h \in \mathbb{R}^M$ with $0 < ||h||_M < \sigma^*$, and $\theta_a \in (0,1)$ it is

$$\sum_{i=1}^M \frac{||h_i||}{||h||_M} \left| \frac{\partial f(a + \theta_a h)}{\partial x_i} - \frac{\partial f(a)}{\partial x_i} \right| < \varepsilon,$$

$$f(a + h) - f(a) = \sum_{i=1}^M \frac{\partial f(a + \theta_a h)}{\partial x_i} h_i.$$

It means that

$$\varepsilon = \sum_{i=1}^M \frac{||h_i||}{||h||_M} \left| \frac{\partial f(a + \theta_a h)}{\partial x_i} - \frac{\partial f(a)}{\partial x_i} \right| =$$

$$= \left| \sum_{i=1}^M \frac{\partial f(a + \theta_a h)}{\partial x_i} h_i - \sum_{i=1}^M \frac{\partial f(a)}{\partial x_i} \frac{h_i}{||h||_M} \right| =$$

$$= \left| \frac{f(a + h) - f(a)}{||h||_M} - \sum_{i=1}^M \frac{\partial f(a)}{\partial x_i} \frac{h_i}{||h||_M} \right| .$$

**Theorem 2.** The operator $N$ is continuous from $H^M_\alpha(\Omega)$ into $H_\alpha(\Omega)$ if and only if the partial derivatives of the first order of the function $f$ are continuous.

**Proof.** Let us suppose, at first, that the partial derivatives of the first order of the function $f$ are continuous. This means that $f$ is a locally lipschitzian func-
tion (if $\sigma \subset \mathbb{R}^M$ is bounded, let $K(\sigma)$ be such a positive number that

$$|f(\xi) - f(\eta)| \leq K(\sigma) \|\xi - \eta\|_M$$

for all $\xi, \eta \in \sigma$). According to Theorem 1 it is

$$\mathcal{H}(H_{\infty}^M(\Omega)) \subset H_{\infty}(\Omega).$$

Now, let us prove the continuity of $\mathcal{H}$ in an arbitrary point $u_0 \in H_{\infty}^M(\Omega)$. Let $\varepsilon > 0$.

Denote

$$W(\Delta) = \{ u \in H_{\infty}^M(\Omega) \mid \|u - u_0\|_{H_{\infty}^M} < \Delta \}$$

for $\Delta > 0$. Then we have

$$\frac{\|u_\Delta(x) - u_\Delta(y)\|_M}{\|x - y\|_N} \leq \Delta < \frac{\|u(x) - u(y)\|_M}{\|x - y\|_N} \leq \frac{\|u_\Delta(x) - u_\Delta(y)\|_M}{\|x - y\|_N} + \Delta$$

for all $u \in W(\Delta), x, y \in \Omega, x \neq y$, and thus there exists $K > 0$ such that

$$\|u(x) - u(y)\|_M \leq K \|x - y\|_N$$

provided $u \in W(\Delta), x, y \in \Omega$. This implies that there exists a bounded set $\sigma \subset \mathbb{R}^M$ such that $u(x) \in \sigma$ for all $u \in W(\Delta)$ and $x \in \Omega$. Put

$$A(u) = \sup_{x \in \Omega} |f(u(x)) - f(u_0(x))| .$$

Obviously

$$A(u) \leq \frac{\varepsilon}{2}$$

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if $u \in W(\Delta) \cap W\left(\frac{e}{2X(0')}\right)$.

We denote $\varepsilon_1 = \frac{e}{8X(0')}$,

$$B(u) = \sup_{x, y \in \Delta \atop x \neq y} \frac{|f(u(x)) - f(u_0(x)) - (f(u(y)) - f(u_0(y)))|}{\|x - y\|_N^\infty},$$

and

$$C(u, x, y) = \frac{|f(u_0(x)) - f(u_0(y))|}{\|u_0(x) - u_0(y)\|_M} \cdot \frac{\|u_0(x) - u_0(y)\|_M}{\|x - y\|_N^\infty},$$

if $u(x) \neq u(y)$, $u_0(x) = u_0(y)$, $x \neq y$.

Thus

$$\|u(x) - u(y) - (u_0(x) - u_0(y))\|_M \leq \varepsilon_1 \|x - y\|_N^\infty$$

for each $x, y \in \Omega$ and $u \in W(\Delta) \cap W\left(\frac{e}{8X(0')}\right)$. It is easy to see that if $u(x) = u(y)$ or $u_0(x) = u_0(y)$ then

$$\frac{|f(u(x)) - f(u_0(x)) - (f(u(y)) - f(u_0(y)))|}{\|x - y\|_N^\infty} \leq \frac{e}{8}. $$

Let $x, y \in \Omega$ and let

$$h'(x, y) = u(x) - u(y) \neq 0, \quad h(x, y) = u_0(x) - u_0(y) \neq 0.$$  

So we obtain

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If $\varepsilon_2 = \frac{\varepsilon}{16X}$, then with respect to the assertion of Lemma 3 there exists $\delta_2 > 0$ such that for each $h', h \in \mathbb{R}^M$, $0 < \| h' \|_M < \delta_2$, $0 < \| h \|_M < \delta_2$ we have

$$\frac{\| f'(x, y) \|_M}{\| h' \|_M} \leq \varepsilon_2,$$

and

$$\frac{\| f(\mu_0(y) + h(x, y)) - f(\mu_0(y)) \|_M}{\| h \|_M} \leq \varepsilon_2.$$

If $\| x - y \|_N < \frac{\delta_2}{X}$ then $\| h'(x, y) \|_M < \delta_2$, $\| h(x, y) \|_M < \delta_2$.

The uniform continuity of partial derivatives of the function $f$ on $\mathcal{O}$ implies the existence of $Q(\mathcal{O}) > 1$ such that

$$\sum_{i=1}^{M} \left| \frac{\partial f(\mu_0(y))}{\partial \xi_i} \right| \leq Q(\mathcal{O})$$

provided $y \in \mathcal{O}$. Let $\varepsilon_3 = \frac{\varepsilon}{16 \cdot X \cdot Q(\mathcal{O})}$ then for
\[u \in W\left(\frac{e}{32.\varepsilon, Q(0)}\right)\text{ we have}
\]
\[\|h'(x,y) - h(x,y)\|_M < \varepsilon_3.
\]

Let \(\varepsilon_4 = \frac{e}{8.\varepsilon, M}\). Using again the uniform continuity of partial derivatives of the function \(f\) on \(\Omega\)' we obtain \(\varepsilon_4 > 0\) so that for each \(u \in H^M(\Omega)\), \(u \in W(\varepsilon_4)\), and for each \(y \in \Omega\) it is
\[
\left| \frac{\partial f(u(y))}{\partial x_i} - \frac{\partial f(u_0(y))}{\partial x_i} \right| < \varepsilon_4.
\]

So
\[
C(u, x, y) \leq
\]
\[
\leq \frac{|f(u(y)) + h'(x,y) - f(u(y))|}{\|h'(x,y)\|_M} \left| \frac{\|h'(x,y)\|_M}{\|u - y\|^{n}_N} - \frac{\|h'(x,y)\|_M}{\|x - y\|^{n}_N} \right| + \frac{|h(x,y)\|_M}{\|u - y\|^{n}_N} \left| \frac{f(u(y)) + h'(x,y) - f(u(y))}{\|h'(x,y)\|_M} \right|
\]
\[
- \frac{|f(u_0(y)) - h'(x,y) - f(u(y))|}{\|h'(x,y)\|_M} \leq \varepsilon_4 \varepsilon_4 + \varepsilon_4 \cdot D(u, x, y),
\]

where
\[
D(u, x, y) = \frac{f(u(y)) + h'(x,y) - f(u(y))}{\|h'(x,y)\|_M} - \frac{f(u_0(y)) + h'(x,y) - f(u(y))}{\|h'(x,y)\|_M}.
\]

The relations (4) imply
\[
D(u, x, y) = 2 \varepsilon_4 + \frac{\sum_{i=1}^{n} \frac{\partial f(u(y))}{\partial x_i} h'_i(x,y)}{\|h'(x,y)\|_M} - \frac{\sum_{i=1}^{n} \frac{\partial f(u_0(y))}{\partial x_i} h'_i(x,y)}{\|h'(x,y)\|_M}.
\]

It means that

\[
(5) \quad C(u, x, y) \leq K(\sigma') e_1 + 2 K e_2 + K M e_4 + 2 K Q(\sigma') e_3 = \frac{\varepsilon}{2}
\]

provided \( u \in \mathcal{W}(\Delta) \cap \mathcal{W} \left( \frac{e}{32 \cdot Q(\sigma') X} \right) \cap \mathcal{W}(\sigma_4) \) and \( x, y \in \Omega \), \( x \neq y \), \( \| x - y \|_N = \frac{\sigma_2}{K} \).

Now, let us suppose that \( \| x - y \|_N \geq \frac{\sigma_2}{K} \).

Then

\[
(6) \quad \left| \frac{\partial f(u(x))}{\partial x} - \frac{\partial f(u_0(x))}{\partial x} - \frac{\partial f(u(y))}{\partial y} - \frac{\partial f(u_0(y))}{\partial y} \right| \leq \frac{\varepsilon}{2}
\]
From (3), (5) and (6) we have: for a given $\varepsilon > 0$ there exists $\sigma' > 0$ such that if $u \in W(\Delta) \cap W(\sigma') \cap W\left(\frac{\varepsilon}{32K.\mathcal{Q}(0')}\right)$ then

$$\| f \ast u - f \ast u_0 \|_{H_\infty} < \varepsilon$$

which is nothing else than the continuity of $\mathcal{N}$.

Now, let us suppose that the operator $\mathcal{N}$ is continuous. Let $\xi_0 \in \mathbb{R}^M$, $\eta_0 \in \Omega$ be fixed. Define

$$u_0(x) = \| x - y_0 \|_N \cdot j_1 + \xi_0, \quad x \in \Omega,$$

$$u_h(x) = \| x - y_0 \|_N \cdot j_1 + \xi_0 + h,$$

where $h \in \mathbb{R}^M$, $x \in \Omega$ and $j_1 = [1, 0, \ldots, 0] \in \mathbb{R}^M$. From the continuity of $\mathcal{N}$ we have: For a given $\varepsilon > 0$ there exists $\sigma' > 0$ such that

$$\sup_{x, y \in \Omega, x \neq y} \left| \frac{f(u_h(x)) - f(u_h(y)) - (f(u_0(x)) - f(u_0(y)))}{\| x - y \|_N} \right| < \varepsilon$$

provided $h \in \mathbb{R}^M$, $\| h \|_M < \sigma'$. Putting $y = y_0$ we obtain
Let $p = \{ x \in \mathbb{R}^M \mid x = [t,0,...,0] + \xi_0, t \in \mathbb{R} \}$. The restriction of the function $f$ on the straight line $p$ is absolutely continuous, for it is locally lipschitzian on $p$ (see Theorem 1). Thus the partial derivative $\frac{\partial f}{\partial \xi_1}$ exists and it is finite almost everywhere. Let us suppose, now, that there exists $\xi_0 \in p$ such that $\frac{\partial f}{\partial \xi_1} (\xi_0)$ does not exist, i.e.,

$$
L = \lim_{t \to 0} \frac{f(\xi_0 + t \xi_1) - f(\xi_0)}{t} > 0
$$

Denote by $\{ \tau_n \}, \{ t_n \}$ the sequences of real numbers with the following properties:

$$
\lim_{n \to \infty} \tau_n = 0, \quad \lim_{n \to \infty} t_n = 0
$$

and

$$
L = \lim_{m \to \infty} \frac{f(\xi_0 + t_m \xi_1) - f(\xi_0)}{t_m}
$$

Let $x_n, y_n \in \Omega$ be such that
\[ \| x_n - y_0 \|_N = t_n, \quad \| y_n - y_0 \|_N = r_n. \]

Moreover, let \( \{ h_n \} \) be such a sequence that \( h_m \in \mathbb{R} \)
with \( \lim_{m \to \infty} h_m = 0 \) and \( \frac{\partial f(\xi_0 + h_m \xi_1)}{\partial \xi_1} \) exists for
all \( m \in \mathbb{N} \). Denote \( \epsilon = \frac{1 - L}{2} \) and, substituting in
(7) \( h = h_m j_1 \), \( x = x_n \), \( \epsilon = \epsilon \) or \( x = y_n \), \( h = h_m j_1 \), \( \epsilon = \epsilon \), we return with
\[ \left| \frac{f(\xi_0 + t_m \xi_1 + h_m \xi_1) - f(\xi_0 + h_m \xi_1)}{t_m} \right| < \epsilon \]
(8)
\[ \left| \frac{f(\xi_0 + t_m \xi_1) - f(\xi_0)}{t_m} \right| < \epsilon \]
(9)
for sufficiently large \( m, n \in \mathbb{N} \). Setting \( n \to \infty \) we
obtain from (8), (9)
\[ \left| \frac{\partial f(\xi_0 + h_m \xi_1)}{\partial \xi_1} - \bar{L} \right| \leq \epsilon, \]
(8')
\[ \left| \frac{\partial f(\xi_0 + h_m \xi_1)}{\partial \xi_1} \right| + \left| \frac{\partial f(\xi_0 + h_m \xi_1)}{\partial \xi_1} - \bar{L} \right| \leq \epsilon. \]
(9')
So the inequalities
\[ 2 \epsilon < |L - \bar{L}| \leq \left| \frac{\partial f(\xi_0 + h_m \xi_1)}{\partial \xi_1} \right| + \left| \frac{\partial f(\xi_0 + h_m \xi_1)}{\partial \xi_1} - \bar{L} \right| \leq 2 \epsilon \]
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are valid for sufficiently large \( m \). This is a contradiction.

Thus \( \frac{\partial f}{\partial x_1}(\xi_0) \) exists for arbitrary \( \xi_0 \in \mathbb{R}^M \).

The continuity of \( \frac{\partial f}{\partial x_1} \) follows easily from (7), letting \( x \) tend to \( y_0 \).

The proof of the existence and continuity of \( \frac{\partial f}{\partial x_1} \) is analogous.

4. Remarks.

A. Let us consider \( 0 < \beta \leq \alpha \leq 1 \). Then the operator \( \mathcal{N} \) maps \( H^M_{\alpha}(\Omega) \) into \( H^M_{\beta}(\Omega) \) if and only if \( f \) is such a function that for each bounded nonempty subset \( \Omega' \) of \( \mathbb{R}^M \) there exists \( K(\Omega') > 0 \) such that

\[
|f(\xi) - f(\eta)| \leq K(\Omega') \|\xi - \eta\|_M^{\alpha}
\]

for each \( \xi, \eta \in \Omega' \). The proof is analogous to that of Theorem 1.

It seems that the necessary and sufficient conditions upon \( f \) for \( \mathcal{N} \) to be continuous remain to be an open problem.

B. If \( 0 < \alpha < \beta \leq 1 \) then \( \mathcal{N} \) maps \( H^M_{\alpha}(\Omega) \) into \( H^M_{\beta}(\Omega) \) if and only if \( f \) is a constant function.
us suppose $\xi_0 \in \mathbb{R}^M$. Denote

$$u(x) = \|x - y_0\|_M^{\infty} \star j_1 + \xi_0, \quad x \in \Omega$$

(as in the proof of Theorem 2). Obviously $u \in H_\infty^M(\Omega)$ and thus

$$\sup_{\frac{\|x - y_0\|^{\alpha}}{x \in \Omega}} \frac{|f(u(x)) - f(u(y_0))|}{\|x - y_0\|^{\beta}} = L < +\infty.$$ 

Denoting $t = \|x - y_0\|_N$ we obtain

$$Lt^{\frac{A}{N} - 1} \geq \left| \frac{f(\xi_0 + t\partial_1) - f(\xi_0)}{t} \right|$$

and so the right hand side derivative of $t \mapsto f(\xi_0 + t\partial_1)$ at $t = 0$ is zero. Similarly the same is valid for the left hand side derivative.

Thus $\frac{\partial f}{\partial \xi_0}(\xi_0) = 0$ and analogously $\frac{\partial f}{\partial \xi_2}(\xi_0) = \ldots = \frac{\partial f}{\partial \xi_M}(\xi_0) = 0$.

C. The investigation of the same problems as in Theorems 1 and 2 for the operator

$$u(x) \mapsto f(x, u(x)),$$

where $f(x, \xi)$ is a given function on $\Omega \times \mathbb{R}^M$, was without success.

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References:


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