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SOME APPLICATIONS OF THE COINCIDENCE DEGREE FOR SET - CONTRACTIONS TO FUNCTIONAL DIFFERENTIAL EQUATIONS OF NEUTRAL TYPE

Georg HETZER, Aachen [1)]

Abstract: We consider the existence of periodic solutions of the neutral functional differential equation $\mathring{x}(t) = f(t, x_t, \mathring{x}_t)$. Basic for the proof of our assertions are two coincidence theorems for an operator equation in Banach spaces, which can be deduced by a coincidence degree theory for set-contractions, given in [6].

Key-words: Coincidence degree, set-contractions, periodic solutions of neutral functional differential equations, alternative problem.

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Introduction. Many problems, concerning nonlinear ordinary partial or functional differential equations, can be reduced to the study of the operator equation $Lx = Nx$, where L is a noninvertible linear map, and N is nonlinear. In respect to partial differential equations, we mention the Landesman-Lazer approach, used e.g. in [7],[17], [3] or [4]. Instead of using Schauder's fixed point theorem

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1) I may thank Professor J. Mawhin for referring me to [5].

in order to ensure the existence of a solution of $Lx = Nx$, J. Mawhin derives a so-called coincidence degree for completely continuous N from the Leray-Schauder degree and establishes a degree continuation theorem for it, which can be applied for solving the above operator equation. For this approach and some of its applications we refer to [8],[9],[10] or [5]. Finally, in [6] we extend the coincidence degree to the situation, where N is a set-contraction.

A significant case for the use of set-contractions is the neutral functional differential equation:

$$(*) \qquad \dot{x}(t) = f(t,x_t,\dot{x}_t) \quad .$$

We refer e.g. to: [14],[15]. Working with completely continuous nonlinearity, Hale and Mawhin ([5]) must restrict themselves to the special case: $f(t,x_t,\dot{x}_t) = \frac{d}{dt} f_1(t,x_t) + f_2(t,x_t)$, where f_1 is linear in x_t , when they study the existence of periodic solutions of $(*)$.

The purpose of this paper is to show that Mawhin's approach can be applied to the general situation, using the coincidence degree for set-contractions.

In Section 1 we mention the needed abstract results of [6] in a somewhat modified form. Then in 2 we reduce the existence of periodic solutions of $(*)$ to:

1. Search an "a priori bound" for f .

2. Determine the Brouwer degree for a special map.

Finally in Section 3 we consider the quasibounded case.

1. Here we collect the later needed abstract results. First we recall some definitions:

Definition 1. Let Y be a metric space and $P(Y)$ the power set of Y. The set-measure of noncompactness $\gamma: P(Y) \longrightarrow \mathbb{R}^+ \cup \{\infty\}$ is defined by:

$$\gamma(M) = \inf\{\varepsilon \mid \varepsilon > 0 , \bigvee_{m\in\mathbb{N}} \bigvee_{(D_1,\dots,D_m)\in P(Y)^m} (\bigwedge_{1\le i\le m} \operatorname{diam}(D_i) \le \varepsilon \wedge \bigcup_{1\le i\le m} D_i \supseteq M)\} .$$

Definition 2. Assume that Y_1, Y_2 are metric spaces, $k \in \mathbb{R}^+$ and $f: Y_1 \longrightarrow Y_2$.

(a) f is called completely continuous: $\Longleftrightarrow$

$$\bigwedge_{B\subseteq Y_1} B \text{ bounded} \Longrightarrow \overline{f(B)} \text{ compact.}$$

(b) f is called k set-contraction: $\Longleftrightarrow$

$$\bigwedge_{B\subseteq Y_1} \gamma(f(B)) \le k\,\gamma(B) .$$

If X, Y are Banach spaces (over $\mathbb{R}$), D a linear subspace of X, and $L: D \longrightarrow Y$ linear, then $\operatorname{Ker}(L)$ denotes the kernel of L and $R(L)$ the range of L. We say that L is a Fredholm operator, if $\alpha(L) := \dim(\operatorname{Ker}(L)) < \infty$, $\beta(L) := \dim(Y/R(L)) < \infty$, L is closed and $R(L)$ is closed. We set: $\operatorname{ind}(L) := \alpha(L) - \beta(L)$. In [6] we have shown that each Fredholm operator L satisfies: $l(L) := \sup\{r \mid r \in \mathbb{R}^+, \bigwedge_{B\subseteq D} \gamma(B) < \infty \Longrightarrow r\gamma(B) \le \gamma(L(B))\} > 0$.

Now we can introduce the following assumptions:

(a) X, Y Banach spaces, D a linear subspace of X,
$\Omega \subseteq X$ open and bounded, $D \cap \Omega \neq \emptyset$,

(b) $L: D \longrightarrow Y$ a Fredholm operator with $\operatorname{ind}(L) = 0$,

(c) $k < l(L)$, $N: \overline{\Omega} \longrightarrow Y$ a k set-contraction.

Since $\operatorname{ind}(L) = 0$, there exist continuous projectors $P: X \longrightarrow X$ and $Q: Y \longrightarrow Y$ with: $R(P) = \operatorname{Ker}(L)$ and $\operatorname{Ker}(Q) = R(L)$ and a linear isomorphism $J: R(Q) \longrightarrow \operatorname{Ker}(L)$.

Further let deg mean the generalized Brouwer degree for finite dimensional vector spaces. Then, using the coincidence degree for set-contractions, we state a theorem in [6] which contains as a special case the following degree continuation result:

<u>Theorem 1</u>. Let (a) - (c) be fulfilled, P, Q and J like above, and assume furthermore:

(1) $\bigwedge_{\lambda \in (0,1)} \bigwedge_{x \in D \cap \partial\Omega} Lx \neq \lambda Nx$,

(2) $\bigwedge_{x \in \operatorname{Ker}(L) \cap \partial\Omega} J \circ Q \circ N(x) \neq 0$,

(3) $\deg(J \circ Q \circ N|_{\operatorname{Ker}(L) \cap \overline{\Omega}}, \operatorname{Ker}(L) \cap \Omega, 0) \neq 0$.

Then there exists an x D with: $Lx = Nx$.

As usual, we set $\deg(J \circ Q \circ N|_{\operatorname{Ker}(L) \cap \overline{\Omega}}, \operatorname{Ker}(L) \cap \Omega, 0) = 1$, if $\operatorname{Ker}(L) = \{0\}$. Before stating the second, later needed abstract result, we recall the definition of a quasi-bounded operator:

Definition 3. Let X, Y be Banach spaces, $F: X \longrightarrow Y$ continuous. F is called quasibounded: $\Longleftrightarrow$

$$|||F||| := \inf_{0<\rho<\infty} \left(\sup\left\{\frac{\|F(x)\|}{\|x\|} \,\middle|\, x \in X, \|x\| \geq \rho\right\}\right) < \infty . \quad |||\ |||$$

is said quasinorm.

From Theorem 1 we obtain like Mawhin in [9] for the case, where N is completely continuous:

Theorem 2. Let (a) - (c) be fulfilled, P, Q, J like above, $\Omega = X$ and K_P the pseudo-inverse of L, associated to P (i.e. $K_P := L|_{D \cap (I - P)(X)})^{-1})$. Furthermore we assume, that there are $\alpha \geq 0$ and $\beta > 0$ with:

(i) $K_P \circ (I - Q) \circ N$ quasibounded,

(ii) $\bigwedge_{x \in X} Q \circ Nx = 0 \Longrightarrow \|Px\| < \alpha \|(I - P)x\| + \beta$,

(iii) $(1 + \alpha) |||K_P \circ (I - Q) \circ N||| < 1$,

(iv) $\deg (J \circ Q \circ N)|_{\mathrm{Ker}(L) \cap B(\beta)}$,

$\mathring{B}(\beta) \cap \mathrm{Ker}(L), 0) \neq 0$,

where $B(\beta) := \{x \mid x \in X, \|x\| \leq \beta\}$.

Then $R(L - N) \supseteq R(L)$.

2. We introduce the following notations:

Let $n \in \mathbb{N}$, $M \subseteq \mathbb{R}$ and $\|\ \|$ a norm of $\mathbb{R}^n$, then $C(M, \mathbb{R}^n)$ is the space of bounded continuous functions from M in $\mathbb{R}^n$, $\|\ \|_\infty$ the supremum norm. For $a > 0$, $t \in \mathbb{R}$ and

$x \in C(\mathbb{R}, \mathbb{R}^n)$ we define $x_t \in C([-a,0], \mathbb{R}^n)$ by: $x_t(\tau) := x(t+\tau)$ for $\tau \in [-a,0]$. If $x \in C(\mathbb{R}, \mathbb{R}^n)$ is differentiable, we write $\dot{x}$ for the derivate of x and $\dot{x}_t$ for the function, given by $\dot{x}_t(\tau) := \dot{x}(t+\tau)$ with $\tau \in [-a,0]$. Finally let $C^1(\mathbb{R}, \mathbb{R}^n)$ denote the subspace of continuously differentiable functions of $C(\mathbb{R}, \mathbb{R}^n)$ and $B_r := \{Z \mid Z \in \mathbb{R}^n, |Z| \leq r\}$ for $r \in \mathbb{R}^+$. We consider the neutral functional differential equation: $\dot{x}(t) = f(t, x_t, \dot{x}_t)$, where $x \in C^1(\mathbb{R}, \mathbb{R}^n)$.

<u>Theorem 3</u>. <u>Assumptions</u>: $n \in \mathbb{N}$, $a > 0$, $f: \mathbb{R} \times C([-a,0], \mathbb{R}^n) \times C([-a,0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ 1-periodic in the first argument, uniformly continuous on bounded sets.

Let $0 \leq k < 1$ with:

(I) $$\bigwedge_{t \in \mathbb{R}} \bigwedge_{u, v_1, v_2 \in C([-a,0], \mathbb{R}^n)} |f(t, u, v_1) - f(t, u, v_2)| \leq k \| v_1 - v_2 \|_\infty .$$

Further there exists an $r > 0$ with:

(II) $$\bigwedge_{x \in C^1(\mathbb{R}, \mathbb{R}^n)} \Big(x \text{ 1-periodic} \wedge \bigvee_{\lambda \in (0,1)} \dot{x}(t) = \lambda f(t, x_t, \dot{x}_t)\Big) \Longrightarrow \max\{\|x\|_\infty, \|\dot{x}\|_\infty\} \neq r$$

(III) $\deg(g|_{B_r}, \mathring{B}_r, 0) \neq 0$, where $g: B_r \longrightarrow \mathbb{R}^n$ is defined by: $g(u) := \int_0^1 f(t, u, 0)\, dt$ for $u \in B_r$ [2].

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2) We identify $u \in \mathbb{R}^n$ and the function: $t \longmapsto u$ for $t \in \mathbb{R}$.

Assertion: There is a continuously differentiable, 1 - periodic function x with: $\dot{x}(t) = f(t, x_t, \dot{x}_t)$ for $t \in \mathbb{R}$.

Proof. We realize the hypotheses of Theorem 1. We set:

$X: = \{x \mid x \in C^1 (\mathbb{R}, \mathbb{R}^n), \bigwedge_{t \in \mathbb{R}} x(t + 1) = x(t)\}$ and

$Y: = \{y \mid y \in C (\mathbb{R}, \mathbb{R}^n), \bigwedge_{t \in \mathbb{R}} y(t + 1) = y(t)\}$. Define $\| \; \| : X \longrightarrow \mathbb{R}^+$ by $\| x \| : = \max \{\| x \|_\infty, \| \dot{x} \|_\infty\}$ and $L: X \longrightarrow Y$ by $Lx: = \dot{x}$. Then we have: $\mathrm{Ker}(L) = \{x \mid x \in X , x$ constant $\}$, hence $\dim(\mathrm{Ker}(L)) = n$. Furthermore we obtain: $z \in R(L) \Longleftrightarrow \int_0^1 z(t)\, dt = 0$, thus $\dim (Y/R(L)) = n$.

Additionally, L is continuous in respect to $\| \; \|$ on X and $\| \; \|_\infty$ on Y and therefore a Fredholm operator with $\mathrm{ind}(L) = 0$. $P: X \longrightarrow X$, defined by $Px: = x(0)$, and $Q: Y \longrightarrow Y$, given by $(Qy)(t): = \int_0^1 y(\tau)\, d\tau$, are projectors with: $R(P) = \mathrm{Ker}(L)$ and $\mathrm{Ker}(Q) = R(L)$. Since the pseudo-inverse K_P of L associated to P , given by $(K_p y)(t) = \int_0^t y(\tau)\, d\tau$ for $y \in R(L)$, has a norm lower than 1 , we obtain: $l(L) \geq 1$.

Now set $\Omega : = \{x \mid x \in X , \| x \| < r\}$ and $(Nx)(t): : = f(t, x_t, \dot{x}_t)$ for $x \in \overline{\Omega}$. The 1 - periodicity of f in the first argument and of x ensures that $Nx \in Y$.

We show: N is uniformly continuous.

Let $\varepsilon > 0$. The uniform continuity of f on bounded sets involves the existence of a $\delta > 0$ with: For all

$u_1, u_2, v_1, v_2 \in C_r := \{u \mid u \in C([-a,0], R^n),$ $\|u\|_\infty \leq r\}$, $\|u_1 - u_2\|_\infty \leq \delta$ and $\|v_1 - v_2\|_\infty \leq \delta$ imply: $|f(t, u_1, v_1) - f(t, u_2, v_2)| \leq \varepsilon$. Let now $x, y \in \overline{\Omega}$ with $\|x - y\| \leq \delta$, then we have:

$$\bigwedge_{t \in \mathbb{R}} \|x_t - y_t\|_\infty \leq \delta \wedge \|\mathring{x}_t - \mathring{y}_t\|_\infty \leq \delta \wedge x_t, \mathring{x}_t, y_t, \mathring{y}_t \in C_r$$

hence: $\bigwedge_{t \in \mathbb{R}} |f(t, x_t, \mathring{x}_t) - f(t, y_t, \mathring{y}_t)| \leq \varepsilon$, thus: $\|Nx - Ny\|_\infty \leq \varepsilon$.

If $z \in X$, we define: $F_z : \overline{\Omega} \longrightarrow Y$ by:

$(F_z x)(t) := f(t, x_t, \mathring{z}_t)$ for $x \in \overline{\Omega}$.

One obtains completely analogous to the uniform continuity of N: $\{F_z \mid z \in X\}$ is uniformly equicontinuous on $\overline{\Omega}$.

Now we can prove: N is a k set-contraction. Let γ respectively γ_∞ be the set-measure of noncompactness on $\overline{\Omega}$ respectively Y according to $\| \ \|$ respectively $\| \ \|_\infty$, and let diam respectively diam_∞ be the symbol for the diameter, taken by $\| \ \|$ respectively $\| \ \|_\infty$. We show that for $A \subseteq \Omega$ $\gamma_\infty(N(A)) \leq k\,\gamma(A)$. Set $\eta := \gamma(A)$. Let $\varepsilon > 0$, then there exist $B_1, \ldots, B_m \subseteq A$ with: $\bigwedge_{1 \leq j \leq m} \text{diam}(B_j) \leq \eta + \varepsilon/2$ and $\bigcup_{1 \leq j \leq m} B_j = A$.

Since $\sup\{\|\mathring{x}\|_\infty \mid x \in A\} \leq r$, Ascoli's theorem implies that $\overline{A}$ is compact in respect to $\| \ \|_\infty$. Then $\overline{F_z(A)}$ is compact for each $z \in X$. Therefore the uniform equicontinuity of the set $\{F_z \mid z \in X\}$ implies: For each $j \in \{1, \ldots, m\}$ there exist $S_1^j, \ldots, S_{n(j)}^j \subseteq B_j$, such that for each $z \in X$ is fulfilled: $\bigwedge_{1 \leq i \leq n(j)} \operatorname{diam}_{\infty} (F_z(S_i^j)) \leq \varepsilon/2$

and $\bigcup_{1 \leq i \leq n(j)} S_i^j = B_j$.

Now we show: $\bigwedge_{1 \leq j \leq m} \bigwedge_{1 \leq i \leq n(j)} \operatorname{diam}_{\infty} (N(S_i^j)) \leq k \eta + \varepsilon$.

Let $j \in \{1, \dots, m\}$, $i \in \{1, \dots, n(j)\}$ and $x, y \in S_i^j$:

$$\| Nx - Ny \|_{\infty} = \| Nx - F_y x \|_{\infty} + \| F_y x - Ny \|_{\infty}$$

$$\begin{aligned} \| Nx - F_y x \|_{\infty} &= \sup_{t \in \mathbb{R}} | f(t, x_t, \mathring{x}_t) - f(t, x_t, \mathring{y}_t) | \\ &\leq k \ \| \mathring{x} - \mathring{y} \|_{\infty} \quad \text{(see (I))} \\ &\leq k \ (\eta + \varepsilon/2) \ (x, y \in B_j) \\ &\leq k \eta + \varepsilon/2 \end{aligned}$$

$\| F_y x - Ny \|_{\infty} = \| F_y x - F_y y \|_{\infty} \leq \varepsilon/2$, since $x, y \in S_i^j$.

Finally we obtain: $\| Nx - Ny \|_{\infty} \leq k \eta + \varepsilon$ for $x, y \in S_i^j$.

The last step of the proof is the realization of (1) - (3) of Theorem 1.

(1) is a direct consequence of Condition (II).

(2) and (3): If $x \in \operatorname{Ker}(L) \cap \overline{\Omega}$, then x is constant and $\| x \|_{\infty} \leq r$, hence:

$((Q \circ N)x) (\tau) = \int_0^1 f(t, x(0), 0) \, dt = g(x(0))$. Thus (III) implies (2) and (3) for every isomorphism J .

Then Theorem 1 ensures the existence of an $x \in \overline{\Omega}$ with $\mathring{x}(t) = f(t, x_t, \mathring{x}_t)$, which completes the proof.

It may be useful to discuss the conditions (I) - (III):

__Remarks__. (1) Obviously an extension of the scalar equation

$\dot{x}(t) = k\,\dot{x}(t) + C$ $(0 \leq k < 1$, $C \neq 0)$ to $k = 1$ is impossible. Therefore a condition like (I) is necessary for the situation, treated in the above theorem.

(2) Condition (II) can be removed in special cases by any a priori bound estimation (e.g. [5]). One observes that (II) is only used, to ensure (1) of Theorem 1.

(3) (III) contains some assumptions, well known in the theory of ordinary differential equations, if one uses e.g. the Borsuk property of the Brouwer degree or the Poincaré-Bohl theorem. Such conditions are:

$$\bigwedge_{t \in \mathbb{R}} \; \bigwedge_{u \in \mathbb{R}^n} \; \bigwedge_{\lambda \in \mathbb{R}^+} \left(|u| = r \Longrightarrow f(t, u, 0) \neq \lambda f(t, -u, 0) \right)$$

or in the special case, where $\| \;$ means the Euclidean norm and $\langle \, , \rangle$ the scalar product of the $\mathbb{R}^n$:

$$\bigwedge_{t \in \mathbb{R}} \; \bigwedge_{u \in \mathbb{R}^n} \left(|u| = r \Longrightarrow \langle u, f(t, u, 0) \rangle > 0 \right) .$$

(4) Finally let us remark that the introduction of guiding functions probably leads to sharper results (e.g. [10] for the special case, where f is independent of $\dot{x}_t$) . But that is not a theme of this paper.

We end this section with a theorem which is related to the results in [1] and [14].

__Theorem 4__. Let $n \in \mathbb{N}$, $f: R \times R^n \times R^n \times R^n \longrightarrow R^n$ be continuous,

1 - periodic in the first argument, $\vartheta, \varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous,
1 - periodic and $0 \leq k < 1$ with:

$$(I') \bigwedge_{t \in \mathbb{R}} \bigwedge_{u,v,w_1,w_2 \in \mathbb{R}^n} |f(t, u, v, w_1) - f(t, u, v, w_2)| \leq k\,|w_1 - w_2| \ .$$

Assume further that there exists an $r > 0$ with:

(II') For each $\lambda \in (0,1)$, the equation $\dot{x}(t) = \lambda f(t, x(t), x(\vartheta(t)), \dot{x}(\varphi(t))$ has no 1-periodic, continuously differentiable solution z with: $\max \{ \| z \|_\infty, \| \dot{z} \|_\infty \} = r$.

(III') $\deg(g|_{B_r}, \mathring{B}_r, 0) \neq 0$, where $g(u) := \int_0^1 f(t, u, u, 0)\, dt$ for $u \in B_r$.

Then there exists a continuously differentiable, 1 - periodic $x \in C(\mathbb{R}, \mathbb{R}^n)$ with:

$$\bigwedge_{t \in \mathbb{R}} \dot{x}(t) = f(t, x(t), x(\vartheta(t)), \dot{x}(\varphi(t))) \ .$$

The proof is analogous to that of Theorem 3 and therefore is omitted here. The remarks in respect to (I) - (III) of Theorem 3 can be transferred to (I') - (III') here.

3. In this section we consider the quasibounded case, using Theorem 2. Two approaches are possible. The first one depends on the substitution of the map $L: x \longmapsto \dot{x}$ by an one-to-one Fredholm operator with index 0 (see [5] for the

case, where N is completely continuous). Theorem 5 is treating this fact. The second approach is to work with the above L, but then the assumptions (ii) - (iv) of Theorem 2 exclude a convenient abstract formulation. Therefore we only give an example for it.

Under a linearity assumption for g we consider the following equation:

$$(*)\qquad \mathring{x}(t) - g(t, x_t, \mathring{x}_t) = f(t, x_t, \mathring{x}_t) \quad .$$

We state:

Theorem 5. Assumptions: $n \in \mathbb{N}$, $a > 0$,

$f, g: \mathbb{R} \times C([-a, 0], R^n) \times C([-a,0], \mathbb{R}^n) \longrightarrow \mathbb{R}^n$

1-periodic in the first argument, uniformly continuous on bounded sets,

$\bigwedge_{t \in \mathbb{R}} g(t, \cdot, \cdot)$ linear [3] , $\tilde{k} \in [0,1)$ with:

$$\bigwedge_{t \in \mathbb{R}} \bigwedge_{u,v_1,v_2 \in C([-a,0],\mathbb{R}^n)} |g(t, u, v_1) - g(t, u, v_2)| \leq \leq \tilde{k} \, \| v_1 - v_2 \|_\infty \quad .$$

Further assume that for each 1-periodic $y \in C(\mathbb{R}, \mathbb{R}^n)$ there exists at most one 1-periodic solution $z \in C^1(\mathbb{R}, \mathbb{R}^n)$ of $\mathring{x}(t) = g(t, x_t, \mathring{x}_t) + y(t)$ $(t \in \mathbb{R})$, and that there is a $c > 0$, independent of y , such that: $c \, \| z \|_\infty \leq \| y \|_\infty$.

Let $k \in [0,c)$ with:

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3) The case, considered in [5], satisfies this linearity condition.

$$\bigwedge_{t \in \mathbb{R}} \bigwedge_{u, v_1, v_2 \in C([-a,0],\mathbb{R}^n)} |f(t, u, v_1) - f(t, u, v_2)| \leq$$
$$\leq k \; \| v_1 - v_2 \|_\infty$$

and:

$$\inf_{0<\rho<\infty} (\sup \{ \|x\|^{-1} \; (\sup_{t \in [0,1]} |f(t, x_t, \dot{x}_t)|) \mid x \in$$
$$\in C^1(\mathbb{R}, \mathbb{R}^n) \; , \; x \; 1\text{-periodic and } \|x\| \geq \rho \}) < c \; .$$

Then: There exists a continuously differentiable 1 - periodic solution of ($*$).

Proof. We set $X := \{x \mid x \in C^1(\mathbb{R},\mathbb{R}^n),\ x \ \text{1-periodic}\}$, $Y := \{y \mid y \in C(\mathbb{R},\mathbb{R}^n),\ y \ \text{1 - periodic}\}$, $L_1 : X \longrightarrow Y$, given by $L_1x := \dot{x}$ and $L_2 : X \longrightarrow Y$, defined by: $(L_2x)(t) = g(t, x_t, \dot{x}_t)$. L_1 and L_2 are continuous linear operators and $\mathrm{ind}(L_1) = 0$ (see proof to Theorem 3). Further, analogous to the proof in Theorem 3, one can show: L_2 is a $\tilde{k}$ set-contraction and then, using Theorem 2 in [6], one receives: $L := L_1 - L_2$ is a Fredholm-operator with $\mathrm{ind}(L) = 0$. By assumption L is injective. Thus the hypotheses (ii) and (iv) of Theorem 2 are satisfied for every N. We set $N: X \longrightarrow Y$ by: $(Nx)(t) = f(t, x_t, \dot{x}_t)$. Using assumption, we have: $c \, \|L^{-1}y\| \leq \|y\|_\infty$ for $y \in Y$. This implies: $\|L^{-1}\|_{\ell} \leq \frac{1}{4}$ 4), hence $l(L) \geq c$. So we get $k < l(L)$ and in regard to the proof of Theorem 3 that N is a k set-contraction. Since L is injective, Conditions

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4) $\|L^{-1}\|_{\ell}$ means the operator norm in respect to $\| \; \|$ on X and $\| \; \|_\infty$ on Y.

(i) and (iii) of Theorem 2 are reduced to $|||L^{-1} \circ N||| < 1$. Now $|||L^{-1} \circ N||| \leq \|L^{-1}\|_{\mathcal{L}} \; |||N||| \leq 1/c \; |||N||| < 1$, using the last assumption of this theorem. So we have realized all hypotheses of Theorem 2 and obtain therefore:

$\bigvee_{x \in X} Lx = Nx$, because $0 \in R(L)$.

The existence of a 1 - periodic solution for the following scalar equation is a simple application of Theorem 5: $\dot{x}(t) - x(t) = f(t, x_t, \dot{x}_t)$, where f is bounded, uniformly continuous on bounded sets and k - Lipschitzian in the third argument with $k < e^{-1}$.

We end the announced example for the second approach.

Let $a > 0$, $g: C([-a,0], \mathbb{R}) \longrightarrow \mathbb{R}$ be a bounded Banach-contraction with constant $k < 1$, $p: \mathbb{R} \longrightarrow \mathbb{R}$ be continuous and odd, and $l \in C(\mathbb{R}, \mathbb{R})$ be 1- periodic. Further assume that $p(u) \longrightarrow \infty$ for $u \longrightarrow \infty$ and $|p(u)| / |u| \longrightarrow 0$ for $|u| \longrightarrow \infty$.

We search 1 - periodic, continuously differentiable solutions of:

(**) $\qquad \dot{x}(t) = l(t) + p(x(t)) + g(\dot{x}_t)$.

Doing this, we realize the hypotheses of Theorem 2. Define X, Y, L, P, Q and K_P as in the proof of Theorem 3. Further set $(Nx)(t) := l(t) + p(x(t)) + g(\dot{x}_t)$. N is a k set-contraction from $(X, \| \|)$ to $(Y, \| \|_\infty)$, because $x \longmapsto l + p \circ x$ is completely continuous and $x \longmapsto g(\dot{x})$ is a Banach-contraction with constant k .

Now we show that (i) - (iv) of Theorem 2 are satisfied. Let M be a bound for $|g|$, then we obtain:

$$\| (I - Q) \circ Nx \|_{\infty} \leq 2(\| l \|_{\infty} + M + \| p \circ x \|_{\infty}) .$$

Thus $|p(u)| / |u| \longrightarrow 0$ for $|u| \longrightarrow \infty$ implies:

$|||K_p \circ (I - Q) \circ N||| = 0$. Hence (i) and for every $\alpha \in \mathbb{R}^+$ (iii) are satisfied.

Since $p(u) \longrightarrow \infty$ for $u \longrightarrow \infty$, we can choose a $\beta > 0$, such that $p(\varrho) \geq M + \| l \|_{\infty} + 1$ for $\varrho \geq \beta$. Now take $\alpha = 1$. Suppose that there is an $x \in X$ with $Q \circ Nx = 0$ and $\| Px \| \geq \| x - Px \| + \beta$. Then we obtain: $|x(0)| \geq \| \mathring{x} \|_{\infty} + \beta$. If $x(0) > 0$ we have: $x(t) \geq x(0) - \| \mathring{x} \|_{\infty} \geq \beta$ for all $t \in \mathbb{R}$, and therefore:

$$\int_0^1 p(x(t))\, dt \geq M + \| l \|_{\infty} + 1 > \left| \int_0^1 l(t) dt + \int_0^1 g(\mathring{x}_t)\, dt \right| ,$$

which is a contradiction to $Q \circ Nx = 0$. If $x(0) < 0$, we obtain $x(t) \leq x(0) + \| \mathring{x} \|_{\infty} = - |x(0))| + \| \mathring{x} \|_{\infty} \leq - \beta$.

Hence:

$$\int_0^1 p(x(t))\, dt \leq -(M + \| l \|_{\infty} + 1) < - \left| \int_0^1 l(t)\, dt + \right.$$
$$\left. + \int_0^1 g(\mathring{x}_t)\, dt \right| ,$$

also a contradiction to $Q \circ N(x) = 0$. Thus (ii) is satisfied.

If $x \in Ker(L)$, then x is a constant function. Thus:

$$\begin{aligned} \| Q \circ N(x) \|_{\infty} &= \left| \int_0^1 l(t)\, dt + \int_0^1 p(x(t))\, dt + g(0) \right| \\ &\geq |p(x(0))| - \| l \|_{\infty} - |g(0)| \\ &> 0 \text{ for } \| x \| = \beta . \end{aligned}$$

Therefore $\deg (J \circ Q \circ N|_{Ker(L) \cap B(\beta)}, \mathring{B}(\beta), 0)$ is defined and different from 0 , using Borsuk's theorem.

Now Theorem 2 implies the existence of an 1 - periodic continuously differentiable solution for (**) .

We remark that the here considered example can be treated by a special case of Theorem 2, which is analogous to Theorem 6.1 in [9].

Results for quasibounded operators, corresponding to the situation in Theorem 4, can be obtained in the same manner. We omit them.

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