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SIMPLY (CO)REFLECTIVE SUBCATEGORIES OF THE CATEGORIES DETERMINED BY POSET-VALUED FUNCTORS

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Abstract: By the criteria in [3] and [4] one sees easily that simply reflective and simply coreflective subcategories (i.e., such (co)reflective subcategories that the (co)-reflections are carried by identities) of the semi-lattice fiberings $\mathcal{O}\!\mathcal{L}_F$ (see 1.3 below, cf. [2]) are again of the form $\mathcal{O}\!\mathcal{L}_G$ for a suitable G . In this note we study the relation of this functor G to the original F . We show that in the reflection case (Theorem 3.4) there is a transformation $\varepsilon : F \longrightarrow G$ and a subtransformation (see 1.4) φ such that $\varepsilon\varphi = 1$, so that G can be considered as a nice factorfunctor of F . In the coreflection case (Theorem 3.7) there is a subtransformation ε and a transformation $\lambda : G \longrightarrow F$ such that $\varepsilon\lambda = 1$. (The φ, λ , resp., are naturally connected with the embedding of $\mathcal{O}\!\mathcal{L}_G$ into $\mathcal{O}\!\mathcal{L}_F$.)

Key words: Simply (co)reflective, generalized lattice fiberings, subtransformation.

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§ 1. Subtransformations

1.1. The category of all sets and mappings is denoted by Set , the category of partially ordered sets and order preserving mappings is denoted by Poset . $\mathcal{D}$ designates the category of partially ordered sets in which every non-void subset has an infimum and of the suprema preserving mappings, CSL is its complete subcategory generated by the complete lattices.

The symbol

$$\mathfrak{X}$$

is used as a variable with values Poset, $\mathcal{D}$, CSL. I.e., the appearance of $\mathfrak{X}$ in a definition or in a statement indicates its applicability for any of the mentioned categories.

1.2. <u>Convention</u>: The partial orderings will be always denoted by the symbol $\leq$. Furthermore, if A, B are partially ordered sets and $f, g: A \longrightarrow B$ mappings, we write

$$f \leq g$$

if $f(a) \leq g(a)$ for every $a \in A$.

1.3. Let $F: \mathrm{Set} \longrightarrow \mathfrak{X}$ be a functor. In accordance with [2] we denote by

$$\mathcal{A}_F$$

the category the objects of which are couples (X,a) with X a set and $a \in F(X)$, the morphisms from (X,a) into (Y,b) being all the triples (a,f,b) with $f: X \longrightarrow Y$ such that $F(f)(a) \leq b$.

$\mathcal{A}_F$ will be considered as a concrete category endowed by the forgetful functor sending (a,f,b) to f .

1.4. Let F , $G: \mathrm{Set} \longrightarrow \mathfrak{X}$ be functors. A subtransformation

$$\tau : F \xrightarrow{\leq} G$$

is a collection of morphisms $\tau = (\tau^X : F(X) \longrightarrow G(X))_{X \in \mathrm{obj}\,\mathrm{Set}}$

such that for every $f\colon X \longrightarrow Y$

$$G(f)\,.\,\tau^X \leq \tau^Y\,.\,F(f)\,.$$

Subtransformations $\tau : F \xrightarrow{\leq} G$ and $\vartheta : G \xrightarrow{\leq} H$ compose in an obvious way. The obtained illegitimate category will be denoted by

$$\mathrm{subtr}\,[\mathrm{Set}, \mathfrak{X}\,]\,.$$

1.5. Consider the categories $\mathcal{A}_F$ with $F\colon \mathrm{Set} \longrightarrow \mathfrak{X}$ and the functors $\Phi : \mathcal{A}_F \longrightarrow \mathcal{A}_G$ such that $V \circ \Phi = U$, where U, V are the natural forgetful functors (see 1.3). The obtained illegitimate category will be denoted by

$$\mathcal{A}_{\mathfrak{X}}\,.$$

1.6. Let $\tau : F \xrightarrow{\leq} G$ be a subtransformation. Define

$$(1)\qquad \begin{aligned} &[\tau]\colon \mathcal{A}_F \longrightarrow \mathcal{A}_G \ \text{ by } [\tau](X,a) = (X, \tau^X(a)) \ \text{ and} \\ &[\tau](a,f,b) = (\tau^X(a), f, \tau^Y(b)) \ \text{ for } f\colon X \longrightarrow Y\,. \end{aligned}$$

(The definition is correct: $G(f)\,\tau^X(a) \leq \tau^Y F(f)(a) \leq \tau^Y(b)$.) Further, let us observe that the category $\mathcal{A}_F$ determines the functor F, since $F(X) = \{a \mid (X,a) \in \mathrm{obj}\,\mathcal{A}_F\}$ and $F(f)(a) = \min\{b \mid (a,f,b) \in \mathrm{morph}\,\mathcal{A}_F\}$. For a functor Φ : $\colon \mathcal{A}_F \longrightarrow \mathcal{A}_G$ such that $V \circ \Phi = U$ define

$$(2)\qquad \langle \Phi \rangle : F \xrightarrow{\leq} G \ \text{ by } \ (X, \langle\Phi\rangle^X(a)) = \Phi\,(X,a)\,.$$

(It is a subtransformation: $\varphi = (a,f,F(f)(a))$ is a morphism, hence $\Phi(\varphi) = (\langle\Phi\rangle^X(a), f, \langle\Phi\rangle^Y F(f)(a))$ is a morphism, so that $G(f)\langle\Phi\rangle^X(a) \leq \langle\Phi\rangle^Y F(f)(a)$.)

After an easy checking the equations

$$[\mathrm{id}] = \mathrm{id}\ ,\ \langle \mathrm{id}\rangle = \mathrm{id},\ [\tau\vartheta] = [\tau]\cdot[\vartheta],\ \langle\Phi.\Psi\rangle =$$
$$= \langle\Phi\rangle.\langle\Psi\rangle,\ \langle[\tau]\rangle = \tau \text{ and } [\langle\Phi\rangle] = \Phi$$

we obtain

<u>Statement</u>: The formulas (1) and (2) establish an isomorphism between $\mathcal{O}\!\mathcal{L}_{\mathfrak{X}}$ and *subtr $[\mathrm{Set}, \mathfrak{X}]$.

1.7. <u>Remark</u>: As a consequence of 1.6 we obtain that $\mathcal{O}\!\mathcal{L}_F$ and $\mathcal{O}\!\mathcal{L}_G$ are equally carried (i.e. there is an isofunctor $\Phi : \mathcal{O}\!\mathcal{L}_F \longrightarrow \mathcal{O}\!\mathcal{L}_G$ with $V \cdot \Phi = U$, cf. 0.2 in [4]) iff F is naturally equivalent to G . (The only point to be checked is that an invertible subtransformation is a transformation and hence a natural equivalence: But if $\tau = \vartheta^{-1}$, we have for $f: X \longrightarrow Y$

$$\tau^Y F(f) = \tau^Y F(f)\,\vartheta^X \tau^X \leq \tau^Y \vartheta^Y\ G(f)\tau^X = G(f)\,\tau^X\ .)$$

§ 2. <u>Concretely adjoint functors</u>

2.1. Let $(\mathcal{A}, U)$, $(\mathcal{B}, V)$ be concrete categories, $L: \mathcal{A} \longrightarrow \mathcal{B}$, $R: \mathcal{B} \longrightarrow \mathcal{A}$ functors such that $V \circ L = U$ and $U \circ R = V$. L (R resp.) is said to be a concretely left (right, resp.) adjoint of R (of L , resp.) if there is a natural equivalence

$$\varkappa^{xy}: \mathcal{B}(L(x)),y) \cong \mathcal{A}(x,R(y))$$

such that $U(\varkappa^{xy}(\varphi)) = V(\varphi)$ for every $\varphi : L(x) \longrightarrow y$.

Remark: The condition of $U(\varkappa(\varphi)) = V(\varphi)$ for every $\varphi : L(x) \longrightarrow y$ is equivalent to a formally weaker one,

$$U(\varkappa^{xL(x)}(1_{L(x)}) = 1_{U(x)} \quad \text{for every } x .$$

Really, we have for $\psi : x \longrightarrow x'$, $\varphi : y' \longrightarrow y$ and $\alpha : L(x') \longrightarrow y'$

$$\varkappa^{xy}(\varphi \cdot \alpha \cdot L(\psi)) = R(\varphi) \cdot \varkappa^{x'y'}(\alpha) \cdot \psi ,$$

so that for $\psi = 1_x$, $y' = L(x)$ and $\alpha = 1_{L(x)}$,

$$\varkappa^{xy}(\varphi) = R(\varphi) \cdot \varkappa^{xL(x)}(1_{L(x)}) .$$

2.3. Proposition: $L: \mathcal{O}\!\mathcal{L}_F \longrightarrow \mathcal{O}\!\mathcal{L}_G$ is a concretely left adjoint of $R: \mathcal{O}\!\mathcal{L}_G \longrightarrow \mathcal{O}\!\mathcal{L}_F$ iff

$$\langle L \rangle \cdot \langle R \rangle \leq 1 \quad \text{and} \quad \langle R \rangle \cdot \langle L \rangle \geq 1 .$$

Proof: Put $\lambda = \langle L \rangle$, $\wp = \langle R \rangle$. Let L be a concretely left adjoint of R. Since $1_{R(X,a)} = (\wp^X(a), 1_X, \wp^X(a))$ is a morphism, $\varkappa^{-1}(1_{R(X,a)}) = (\lambda^X \wp^X(a), 1_X, a)$ is a morphism, and hence $\lambda^X \wp^X(a) \leq a$. Similarly, using $1_{L(X,b)}$, $b \leq \wp^X \lambda^X(b)$.

On the other hand, let $\lambda\wp \leq 1$ and $1 \leq \wp\lambda$. Take an $f: X \longrightarrow Y$. If $G(f)\lambda^X(a) \leq b$, we have $F(f)(a) \leq F(f)\wp^X\lambda^X(a) \leq \wp^Y G(f)\lambda^X(a) \leq \wp^Y(b)$; if $F(f)(a) \leq \wp^Y(b)$, we have $G(f)\lambda^X(a) \leq \lambda^Y F(f)(a) \leq \lambda^Y\wp^Y(b) \leq b$. Thus, f carries a morphism $L(X,a) \longrightarrow (Y,b)$ iff it carries a morphism $(X,a) \longrightarrow R(Y,b)$.

2.3. <u>Remark</u>: Let us have collections of morphisms $(\lambda^X : G(X) \longrightarrow F(X))_X$ and $(\rho^X : F(X) \longrightarrow G(X))_X$ such that always

$$\lambda^X \rho^X \leq 1 \quad \text{and} \quad \rho^X \lambda^X \geq 1 \ .$$

Then 1) If (λ^X) is a transformation, then (ρ^X) is a subtransformation,

2) If (ρ^X) is a transformation and (λ^X) is a subtransformation, then (λ^X) is a transformation.

Really, in the first case we have

$$F(f)\rho^X \leq \rho^Y \lambda^Y F(f)\rho^X = \rho^Y G(f) \lambda^X \rho^X \leq \rho^Y G(f) \ ,$$

in the second one,

$$\lambda^Y F(f) \leq \lambda^Y F(f) \rho^X \lambda^X = \lambda^Y \rho^Y G(f) \lambda^X \leq G(f) \lambda^X$$

2.4. Following [1], a subcategory $\mathcal{B}$ of a concrete category $(\mathcal{A},U)$ is said to be simply reflective (coreflective, resp.) if the embedding $(\mathcal{B},U \mid \mathcal{B}) \subset (\mathcal{A},U)$ has a concretely left (right, resp.) adjoint. (In other words, if it is (co)reflective and the (co)reflection morphisms are identity carried.)

2.5. <u>Lemma</u>: Every coretraction in $\mathcal{A}_{\mathfrak{X}}$ is a full embedding.

<u>Proof</u>: Let $V \cdot \Phi = U$, $U \cdot \Psi = V$ and $\Psi \Phi = 1$. Let $\varphi : \Phi(a) \longrightarrow \Phi(b)$ be a morphism. We have $\psi = \Psi(\varphi)$: $: a \longrightarrow b$ and $V \Phi \Psi(\varphi) = U \Psi(\varphi) = V(\varphi)$. Since the

forgetful functors are faithful, $\Phi(\psi) = \varphi$.

2.6. By 2.5 and 1.6 we obtain immediately:

<u>Corollary</u>: Let $\Phi : \mathcal{O}\!\mathcal{L}_G \to \mathcal{O}\!\mathcal{L}_F$ be such that $V \cdot \Phi =$ $= U$. If there is a subtransformation λ (ρ , resp.) such that $\lambda \cdot \langle \Phi \rangle = 1$ and $\langle \Phi \rangle \lambda \geq 1$ ($\rho \cdot \langle \Phi \rangle = 1$ and $\langle \Phi \rangle \cdot \rho \leq 1$, resp.) then Φ is an isomorphism onto a simply reflective (coreflective, resp.) subcategory of $\mathcal{O}\!\mathcal{L}_F$, equally carried with $\mathcal{O}\!\mathcal{L}_G$.

§ 3. <u>Simply (co)reflective subcategories of</u> $\mathcal{O}\!\mathcal{L}_F$.

3.1. In this paragraph we will show that there are no other simply (co)reflective subcategories of an $\mathcal{O}\!\mathcal{L}_F$ but those embedded as in 2.6. First, let us make a few observations, actually trivial restatements of the definitions combined with an introduction of a notation which will be used in the sequel.

Let $\mathfrak{K}$ be a simply reflective subcategory of $\mathcal{O}\!\mathcal{L}_F$. Then, for every $a \in F(X)$ we have an $\bar{a} \in F(X)$ such that

1) $a \leq a$,

2) $(X,\bar{a}) \in \mathrm{obj}\, \mathfrak{K}$,

3) If $(Y,b) \in \mathrm{obj}\, \mathfrak{K}$ and if $F)f)(a) \leq b$, then $F(f)(\bar{a}) \leq b$.

Similarly, if $\mathfrak{K}$ is a concretely coreflective subcategory of $\mathcal{O}\!\mathcal{L}_F$, then for every $a \in F(X)$ we have an $\underline{a} \in F(X)$ such that

1^c $\underline{a} \leq a$,

2^c) $(X,\underline{a}) \in \operatorname{obj} \mathfrak{K}$,

3^c) If $(Y,b) \in \operatorname{obj} \mathfrak{K}$ and if $F(f)(b) \leq a$, then $F(f)(b) \leq \underline{a}$.

3.2. By an easy reasoning we obtain

<u>Lemma</u>: a) $\overline{a} = \min \{ b \mid (X,b) \in \operatorname{obj} \mathfrak{K} \ \& \ a \leq b \}$. In particular, $\overline{a} = a$ for $(X,a) \in \operatorname{obj} \mathfrak{K}$.

b) $a \leq b \Longrightarrow \overline{a} \leq \overline{b}$.

c) $F(f)(\overline{a}) \leq \overline{F(f)(a)} = \overline{F(f)(\overline{a})}$.

3.3. Let $\mathfrak{K}$ be simply reflective. Put $G(X) = \{ a \mid (X,a) \in \operatorname{obj} \mathfrak{K} \}$.

We obtain easily

<u>Lemma</u>: a) If a_i $(i \in J)$ are in $G(X)$ and if there is an infimum a of $\{a_i\}$ in $F(X)$, then $a \in G(X)$.

b) If a is a supremum of $\{a_i\}$ in $F(X)$, then $\overline{a}$ is a supremum of $\{\overline{a}_i\}$ in $G(X)$.

3.4. <u>Theorem</u>: Let $\mathfrak{K}$ be a simply reflective subcategory of $\mathfrak{A}_F$, F: Set $\longrightarrow \mathfrak{X}$. Then there is a functor G: : Set $\longrightarrow \mathfrak{X}$, a transformation $\lambda : F \longrightarrow G$ and a subtransformation $\rho : G \longrightarrow F$ such that

(i) $\mathfrak{K} = \mathfrak{A}_G$ and $[\rho] = (\mathfrak{K} \subset \mathfrak{A}_F)$,

(ii) $\lambda\rho = 1$ and $\rho\lambda \geq 1$.

<u>Proof</u>: Put

$(*)$ $G(X) = \{ a \mid (X,a) \in \operatorname{obj} \mathfrak{K} \}$, $G(f)(a) = \overline{F(f)(a)}$.

We have

$$a \leq b \Longrightarrow G(f)(a) \leq G(f)(b)$$

by 3.2 b). By 3.2 a), $G(1)(a) = \bar{a} = a$. By 3.2 c) we have

$$G(g)G(f)(a) = \overline{F(g)(\overline{F(f)(a)})} = \overline{F(g)F(f)(a)} = \overline{F(gf)(a)} = G(gf)(a) .$$

Thus, the formulas $(*)$ define a functor $G\colon \mathrm{Set} \longrightarrow \mathrm{Poset}$. Now, let $\mathfrak{X} = \mathfrak{D}$ or $\mathfrak{X} = \mathrm{CSL}$. Then obviously, by 3.3 a), every $G(X)$ is in obj $\mathfrak{D}$ or obj CSL . In any case, every subset with an upper bound has a supremum. Now, let a be a supremum of $\{a_i\}$ in $G(X)$. Thus, $\{a_i\}$ has a supremum b in $F(X)$ and we have by 3.3 b) and 3.2 c), and by 3.3 b) again,

$$G(f)(a) = \overline{F(f)(b)} = \overline{\sup_{F(X)} F(f)(a_i)} = \sup_{G(X)} \overline{F(f)(a_i)} =$$

$$= \sup_{G(X)} G(f)(a_i) .$$

We have

$$F(f)\, \varphi^X (a) = F(f)(a) \leq \overline{F(f)(a)} = G(f)(a) = \varphi^Y G(f)(a) ,$$

$$G(f)\, \lambda^X (a) = G(f)(\bar{a}) = \overline{F(f)(\bar{a})} = \overline{F(f)(a)} = \lambda^Y F(f)(a) ,$$

and

$$\lambda^X \varphi^X (a) = a \quad \text{(by 3.2 a))}, \quad \varphi^X \lambda^X (a) = \bar{a} \geq a .$$

If $\varphi = (a,f,b)$ is a morphism in $\mathfrak{A}_G$, we have $G(f)(a) \leq b$. Hence, $F(f)(a) \leq \overline{F(f)(a)} \leq b$, so that φ is in $\mathfrak{A}_F$. Thus, $\mathfrak{A}_G$ is a subcategory of $\mathfrak{A}_F$. Since obviously obj $\mathfrak{A}_G$ = obj $\mathfrak{K}$ and since, by 2.5, $[\varphi]$ is a full embedding, we obtain $\mathfrak{A}_G = \mathfrak{K}$.

3.5. For the $\underline{a}$ (see 3.1) we obtain easily

Lemma: a) $\underline{a} = \max\{b \mid (X,b) \in \operatorname{obj} \mathfrak{K} \ \& \ b \leq a\}$. In particular, $\underline{a} = a$ for $(X,a) \in \operatorname{obj} \mathfrak{K}$.

b) $a \leq b \implies \underline{a} \leq \underline{b}$.

c) $F(f)(\underline{a}) \leq \underline{F(f)(a)}$.

3.6. Let $\mathfrak{K}$ be simply coreflective. Put $G(X) = \{a \mid (X,a) \in \operatorname{obj} \mathfrak{K}\}$.

We obtain easily

Lemma: a) If a_i $(i \in J)$ are in $G(X)$, and if there is a supremum a of $\{a_i\}$ in $F(X)$ (in $G(X)$, resp.), then $a \in G(X)$ (then a is also a supremum of $\{a_i\}$ in $F(X)$, resp.).

b) If a is an infimum of $\{a_i\}$ in $F(X)$, then $\underline{a}$ is an infimum of $\{\underline{a_i}\}$ in $G(X)$.

3.7. Theorem: Let $\mathfrak{K}$ be a simply coreflective subcategory of $\mathfrak{A}_F$, $F\colon \mathrm{Set} \longrightarrow \mathfrak{X}$. Then there is a functor $G\colon \mathrm{Set} \longrightarrow \mathfrak{X}$, a subtransformation $\varphi : F \xrightarrow{\leq} G$ and a transformation $\lambda : G \longrightarrow F$ such that

(i) $\mathfrak{K} = \mathfrak{A}_G$ and $[\lambda] = (\mathfrak{K} \subset \mathfrak{A}_F)$,

(ii) $\lambda\varphi \leq 1$ and $\varphi\lambda = 1$.

Proof: Since $F(f)(a) \leq F(f)(a)$, we have by 3.5 c) for $(X,a) \in \operatorname{obj} \mathfrak{K}$ $F(f)(a) \leq \underline{F(f)(a)}$, and hence (using also 1^c) from 3.1), for $(X,a) \in \text{obj}\ \mathfrak{K}$ and any $f: X \longrightarrow Y$,

$$(Y, F(f)(a)) \in \text{obj}\ \mathfrak{K} .$$

Thus, we may define a functor G: Set $\longrightarrow$ Poset putting

$$(*) \quad G(X) = \{a \mid (X,a) \in \text{obj}\ \mathfrak{K}\},\ G(f)(a) = F(f)(a) .$$

Now, let $\mathfrak{X} = \mathfrak{D}$ or $\mathfrak{X} = \mathrm{CSL}$. Then, by 3.6 b), every $G(X)$ is in obj $\mathfrak{X}$ by 3.6 a) every $G(f)$ preserves suprema. Thus, G may be regarded as a functor Set $\longrightarrow \mathfrak{X}$.

Define $\varphi : F \xrightarrow{\leq} G$ and $\lambda : G \longrightarrow F$ putting $\varphi^X(a) = \underline{a}$ and $\lambda^X(a) = a$. (We have, by 3.5 c), $G(f)\ \varphi^X(a) = G(f)(\underline{a}) = F(f)\ (\underline{a}) \leq \underline{F(f)(a)} = \varphi^Y F(f)(a)$, and obviously $F(f)\ \lambda^X(a) = F(f)(a) = \lambda^Y G(f)(a))$. We have, by 3.5 a), $\varphi^X \lambda^X(a) = \underline{\lambda^X(a)} = a$ and $\lambda^X \varphi^X(a) = \underline{a} \leq a$.

If $\varphi = (a,f,b)$ is a morphism in $\mathcal{O}\!\ell_G$, we have $F(f)(a) = G(f)(a) \leq b$. Thus, $\mathcal{O}\!\ell_G$ is a subcategory of $\mathcal{O}\!\ell_F$. Since obviously obj $\mathcal{O}\!\ell_G$ = obj $\mathfrak{K}$ and since, by 2.5, $[\lambda]$ is a full embedding, we obtain $\mathcal{O}\!\ell_G = \mathfrak{K}$.

3.8. Remark: By 2.6, 3.4 and 3.7 we see that whenever for subtransformations φ, λ holds $\lambda\varphi = 1$ and $\varphi\lambda \geq 1$ (or, $\lambda\varphi \leq 1$ and $\varphi\lambda = 1$), then λ is a transformation. This, of course, follows easily directly: in the first case we have $\lambda^Y F(f) \leq \lambda^Y F(f)\varphi^X \lambda^X \leq \lambda^Y \varphi^Y G(f)\lambda^X =$ $= G(f)\lambda^X$, in the second one, $\lambda^Y G(f) = \lambda^Y G(f)\varphi^X\lambda^X \leq$ $\leq \lambda^Y \varphi^Y F(f)\lambda^X \leq F(f)\lambda^X$.

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