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## A NOTE ON SUBQUASIVARIETIES OF SOME VARIETIES OF LATTICES

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Abstract: This paper is concerned with varieties of lattices, all subquasivarieties of which are varieties.

Key words: Lattice, variety, quasivariety, primitive lattice.

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V.I. Igošin has shown in [1] that the variety of lattices defined by the inclusion  $a \wedge (b \vee (c \wedge d)) \wedge (c \vee d) \leq b \vee (a \wedge c) \vee (a \wedge d)$  has no subquasivariety which is not a variety. We shall give also some examples of such varieties of lattices.

Given a lattice  $L$ , we denote by  $\mathcal{N}(L)$  the class of all lattices that contain no sublattice isomorphic to  $L$ . Let  $\mathcal{K}$  be a class of lattices. A lattice  $L \in \mathcal{K}$  is called weakly  $\mathcal{K}$ -projective iff  $L$  can be embedded in any lattice in  $\mathcal{K}$  that has a homomorphic image isomorphic to  $L$ . A lattice is said to be primitive ( $\mathcal{K}$ -primitive,  $\mathcal{K}$  is a variety of lattices) if the class  $\mathcal{N}(L) (\mathcal{N}(L) \cap \mathcal{K})$  is a variety. It is easily verified that a non-trivial subdirectly irreducible lattice  $L$  is  $\mathcal{K}$ -primitive if and only if  $L$  is weakly  $\mathcal{K}$ -projective.

**Theorem 1.** Let  $\mathcal{K}$  be a variety of lattices. The following conditions are equivalent.

- (1) Any subquasivariety of  $\mathcal{K}$  is a variety.
- (2) Any non-trivial subdirectly irreducible lattice in  $\mathcal{K}$  is  $\mathcal{K}$ -primitive.
- (3) Any subdirectly irreducible lattice in  $\mathcal{K}$  is weakly  $\mathcal{K}$ -primitive.

**Proof.** Assume (1) and let  $L$  be a non-trivial subdirectly irreducible lattice. The class  $\mathcal{N}(L) \cap \mathcal{K}$  is a subquasivariety of  $\mathcal{K}$  and so by (1), it is a variety, i.e. the condition (2) holds. Evidently, (2) is equivalent to (3). Now suppose (3) and let  $\mathcal{A}$  be a subquasivariety of  $\mathcal{K}$  and let  $\mathcal{B}$  be the variety generated by  $\mathcal{A}$ . We shall show  $\mathcal{A} = \mathcal{B}$ . Since any lattice in  $\mathcal{B}$  is isomorphic to a subdirect product of subdirectly irreducible lattices from  $\mathcal{B}$  and  $\mathcal{A}$  is closed under the formation of products and sublattices, it suffices to prove that all subdirectly irreducible lattices of  $\mathcal{B}$  belong to  $\mathcal{A}$ . Let  $L \in \mathcal{B}$  be subdirectly irreducible. There exists a homomorphism of a lattice  $M \in \mathcal{A}$  onto  $L$  and by (3)  $M$  contains a sublattice isomorphic to  $L$ . Since  $\mathcal{A}$  is closed under sublattices, we have  $L \in \mathcal{A}$ , and this is what we were required to prove.

A class  $\mathcal{K}$  of lattices is called locally finite if any finite subset of any lattice in  $\mathcal{K}$  generates a finite sublattice. If  $\mathcal{A}$  is a set of lattices such that for any positive integer  $n$  there exists a positive integer  $\varphi(n)$

such that any  $n$  elements of any lattice in  $\mathcal{A}$  generate a sublattice of cardinality  $\leq \varphi(n)$ , then  $\mathcal{A}$  generates a locally finite variety (see [5]). Given a class of lattices  $\mathcal{K}$  we shall denote by  $\text{Fin}(\mathcal{K})$  the class of all finite lattices of  $\mathcal{K}$ .

Theorem 2. Let  $\mathcal{K}$  be a locally finite variety of lattices. The following conditions are equivalent.

- (1) Any subquasivariety of  $\mathcal{K}$  is a variety.
- (2) Any non-trivial finite subdirectly irreducible lattice in  $\mathcal{K}$  is  $\mathcal{K}$ -primitive.
- (3) Any finite subdirectly irreducible lattice in  $\mathcal{K}$  is weakly  $\mathcal{K}$ -projective.
- (4) Any finite subdirectly irreducible lattice in  $\mathcal{K}$  is weakly  $\text{Fin}(\mathcal{K})$ -projective.

Proof. It suffices to prove that (4) implies (3). Assume (4) and let  $\mathcal{A}$  be a subquasivariety of  $\mathcal{K}$ . Denote by  $\mathcal{B}$  the subvariety of  $\mathcal{K}$  generated by  $\mathcal{A}$ . Suppose  $\mathcal{A} \not\subseteq \mathcal{B}$ . Then there exists a finitely generated lattice  $L \in \mathcal{B}$  such that  $L \notin \mathcal{A}$ . Since  $\mathcal{K}$  is locally finite,  $L$  is finite. The lattice  $L$  is a homomorphic image of a lattice  $M \in \mathcal{A}$ . We can assume that  $M$  is finitely generated and since  $M \in \mathcal{K}$ , we see that  $M$  is finite.  $L$  is isomorphic to a subdirect product of finite subdirectly irreducible lattices  $A_\iota \in \mathcal{B}$  ( $\iota \in I$ ). So we get that any  $A_\iota$  ( $\iota \in I$ ) is a homomorphic image of  $M$  and by (4)  $M$  contains sublattices isomorphic to  $A_\iota$  ( $\iota \in I$ ).

The class  $\mathcal{A}$  is closed under the formation of sublattices and products and thus we get that all  $A_L$  ( $L \in I$ ) are in  $\mathcal{A}$  and so  $L$  is also in  $\mathcal{A}$ ; a contradiction.

Let  $L$  be a lattice. Define a lattice  $L^*$  in this way:  $L$  is a sublattice of  $L^*$ ,  $L^* \setminus L$  contains exactly three elements  $a, u, v$ ;  $v$  is the smallest element of  $L^*$ ,  $u$  is the greatest element of  $L^*$  and  $a$  is comparable with no element of  $L$ . Given a finite lattice  $L$  we denote by  $L^0$  a lattice which is obtained from  $L$  by adding exactly one element comparable only with the greatest and the smallest element of  $L$ .

Let  $\mathcal{K}$  be a class of lattices. A lattice  $L \in \mathcal{K}$  will be called semi  $\mathcal{K}$ -projective if the following condition holds: whenever  $\varphi$  is a homomorphism of  $A \in \mathcal{K}$  onto  $L$  then there exists a homomorphism  $\psi$  of  $L$  into  $A$  such that  $\varphi \circ \psi(x) = x$  for all  $x \in L$ , i.e.  $\varphi \circ \psi = id_L$ .

Lemma 1. Let  $\mathcal{K}$  be a class of lattices and let  $L \in \mathcal{K}$  and  $L^* \in \mathcal{K}$ . If  $L$  is weakly  $\mathcal{K}$ -projective, then  $L^*$  is weakly  $\mathcal{K}$ -projective. If  $L$  is semi  $\mathcal{K}$ -projective, then  $L^*$  is also semi  $\mathcal{K}$ -projective.

Proof. Let  $\varphi$  be a homomorphism of a lattice  $A \in \mathcal{K}$  onto  $L^*$ . Let  $a \in L^*$  be comparable with no element of  $L$  and denote by  $b$  the smallest and by  $c$  the greatest element of  $L$ . There exist  $a', b', c' \in A$  such that  $\varphi(a') = a$ ,  $\varphi(b') = b$ ,  $\varphi(c') = c$ . Put  $v' = b' \vee a'$ ,  $c'' = (c' \wedge v') \vee b'$ ,  $u' = c'' \wedge a'$  and  $b'' = b' \vee u'$ . One can easily show that  $u' < b'' < c'' < v'$ ,  $c'' \wedge a' = u'$ ,

$b'' \vee a' = v'$  and  $\varphi(c'') = c$ ,  $\varphi(b'') = b$ . Since the interval  $I = \{x \in A; b'' \leq x \leq c''\}$  is mapped by  $\varphi$  onto  $L$ , we have that it contains a sublattice  $L'$  isomorphic to  $L$ . It is easy to verify that the set  $L' \cup \{a', u', v'\}$  forms a sublattice of  $A$  isomorphic to  $L^*$ . If  $L$  is semi  $\mathcal{K}$ -projective, then there exists a homomorphism  $\psi$  of  $L$  into  $I$  such that  $\varphi \circ \psi = id_L$ . Let  $u$  and  $v$  be the greatest and the smallest element of  $L^*$ . Define a mapping  $\bar{\psi}$  of  $L^*$  into  $I$  by  $\bar{\psi}(x) = \psi(x)$  for all  $x \in L$ ,  $\bar{\psi}(u) = u'$ ,  $\bar{\psi}(v) = v'$  and  $\bar{\psi}(a) = a'$ . One can easily show that  $\bar{\psi}$  is a homomorphism of  $L^*$  into  $I$  such that  $\varphi \circ \bar{\psi} = id_{L^*}$ .

Lemma 2. Let  $\mathcal{K}$  be a class of finite lattices and let  $L$  be a semi  $\mathcal{K}$ -projective lattice. If  $L^0$  is in  $\mathcal{K}$ , then  $L^0$  is also semi  $\mathcal{K}$ -projective.

Proof. Let  $\varphi$  be a homomorphism of a lattice  $A \in \mathcal{K}$  onto  $L^0$ . Let  $u$  be the greatest and  $v$  the smallest element of  $L$ . Denote by  $u_0$  the smallest element of  $A$  that is mapped by  $\varphi$  onto  $u$  and by  $v_0$  the greatest element of  $A$  that is mapped by  $\varphi$  onto  $v$ . Let  $b$  be an element in  $A$  such that  $\varphi(b) = a \in L^0 \setminus L$ . The interval  $I = \{x \in A; v_0 \leq x \leq u_0\}$  is mapped by  $\varphi$  onto  $L$  and thus there exists a homomorphism  $\psi$  of  $L$  into  $I$  such that  $\varphi \circ \psi = id_L$ . Define  $b' = (b \vee v_0) \wedge u_0$ . Evidently  $\varphi(b') = a$ . It is easy to show that a mapping  $\bar{\psi}$  of  $L^0$  into  $I$  defined by  $\bar{\psi}(x) = \psi(x)$  for all  $x \in L$  and  $\bar{\psi}(a) = b'$  is a homomorphism of  $L^0$  into  $I$  such that  $\varphi \circ \bar{\psi} = id_{L^0}$ .

The class of all lattices will be denoted by  $\mathbb{L}$  and the class of all finite lattices will be denoted by  $\text{Fin}(\mathbb{L})$ . For any positive integer  $n \geq 3$  we shall denote by  $M_n$  the lattice of dimension 2 and cardinality  $n + 2$ .

Corollary 1. The lattices  $M_n$  are semi  $\text{Fin}(\mathbb{L})$ -projective.

Proof. For any positive integer  $n \geq 3$  the lattice  $M_n$  can be obtained in a finite number of steps from the three element chain by application of  $\circ$ .

Lemma 3. Let  $\mathbb{K}$  be a locally finite variety of lattices generated by a class  $\mathbb{A}$  of lattices. If  $L \in \mathbb{K}$  is a finite subdirectly irreducible lattice, then  $L$  is a homomorphic image of a sublattice of a lattice  $B \in \mathbb{A}$ .

Proof. By [3]  $L$  is a homomorphic image of a sublattice  $C$  of an ultraproduct of lattices from  $\mathbb{A}$ . We can suppose that  $C$  is finitely generated and since  $\mathbb{K}$  is locally finite, we have that  $C$  is finite. The class  $\mathbb{N}(C)$  is closed under the formation of ultraproducts and thus there exists a lattice  $B \in \mathbb{A}$  that contains a sublattice isomorphic to  $C$ .

Theorem 3. Let  $\mathbb{A}$  be a class of lattices such that the following conditions hold:

- (1) The variety  $\mathbb{V}$  generated by  $\mathbb{A}$  is locally finite.
- (2) Any finite subdirectly irreducible lattice which is a homomorphic image of a sublattice of a lattice from  $\mathbb{A}$  is weakly  $\text{Fin}(\mathbb{V})$ -projective.

Then any subquasivariety of  $\mathcal{V}$  is a variety.

Proof. If  $L$  is a finite subdirectly irreducible lattice of  $\mathcal{V}$ , then by Lemma 3 there exists a lattice  $B \in \mathcal{A}$  such that  $L$  is a homomorphic image of a sublattice of  $B$ . By (2)  $L$  is  $\text{Fin}(\mathcal{V})$ -projective. Now, Theorem 3 follows from Theorem 2.

Corollary 2. Let  $\mathcal{M}$  be a finite set of semi  $\text{Fin}(\mathcal{L})$ -projective lattices and let any subdirectly irreducible lattice which is a homomorphic image of a sublattice of a lattice from  $\mathcal{M}$  be semi  $\text{Fin}(\mathcal{L})$ -projective. Let  $\mathcal{N}$  be the set of all lattices which can be obtained from a lattice of  $\mathcal{M}$  in a finite number of steps by applications of  $*$  and  $\circ$ . Then any subquasivariety of the variety  $\mathcal{V}$  generated by  $\mathcal{N}$  is a variety.

Proof. One can easily show that the conditions (1) and (2) hold.

Corollary 3. Let  $\mathcal{M}$  be the class of all lattices that can be obtained in a finite number of steps starting from a lattice  $L_i$  ( $i = 1, 2, \dots, 7$ ) in Fig. 1 by applications  $*$  and  $\circ$ . Then all subquasivarieties of the variety  $\mathcal{V}$  generated by  $\mathcal{M}$  are varieties.

Proof. The lattices  $L_1 - L_6$  are primitive (see [2]) and so they are sublattices of the free lattice and thus  $L_1 - L_6$  are projective (see [6]). The lattice  $L_7 = M_3$  is semi  $\text{Fin}(\mathcal{L})$ -projective by Corollary 1. Now one can easily show that the conditions (1) and (2) of Theorem 3 hold.



**Corollary 4.** (Igošin [11].) All subquasivarieties of the variety  $\mathbb{V}_0$  of lattices defined by the inclusion

$$a \wedge (b \vee (c \wedge d)) \wedge (c \vee d) \leq b \vee (a \wedge c) \vee (a \wedge d)$$

are varieties.

**Proof.**  $\mathbb{V}_0$  is generated by the set of lattices  $\{M_n; 3 \leq n < \omega\}$  (see [4]) and thus we have that  $\mathbb{V}_0$  is a subvariety of the variety  $\mathbb{W}$  in Corollary 3.

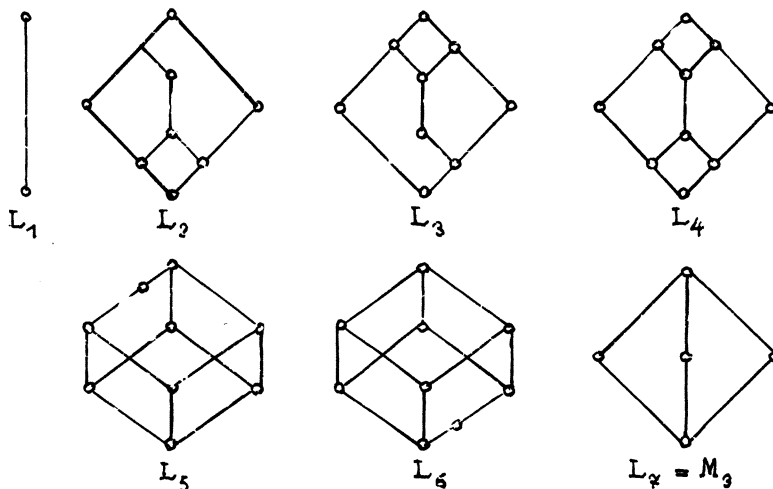


Figure 1

R e f e r e n c e s

- [1] IGOŠIN V.I.: Kvizimnogoobrazija rešetok, *Mat.Zametki* 16(1974),49-56.
- [2] JEŽEK J. and SLAVÍK V.: Some examples of primitive lattices, *Acta Univ.Carol.-Math.et Phys.*14(1973), 3-8.
- [3] JÓNSSON B.: Algebras whose congruence lattices are distributive, *Math.Scand.*21(1967),110-121.
- [4] JÓNSSON B.: Equational classes of lattices, *Math.Scand.* 22(1968),187-196.
- [5] MALCEV A.I.: *Algebraičeskie sistemy*, M.,1970.
- [6] KOSTINSKY A.: Projective lattices and bounded homomorphisms, *Pacific J.Math.*40(1972),111-119.

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