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ON UNIFORM SPACES

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Abstract: Cardinal (particularly proximal), distal and coz refinements of the category of uniform spaces are studied.

Key words: cardinal reflection, distal spaces, partitions of unity, $\mathcal{K}$-fine, $\mathcal{K}$-coarse, products of proximally fine.

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The publication of the seminar notes of my Seminar Uniform Spaces 1974-1975, to appear in the series Publications of Mathematical Institute of ČSAV, No 1, has been delayed. This is to announce basic results which admit simple statements. For details and further results the reader is referred to the seminar notes which contain 15 notes by the author, J. Hejcman, M. Hušek, M. Kosina, V. Kůrková, J. Pelant, P. Pták, J. Reiterman, V. Rödl, J. Vilímovský, and M. Zahradník. The starting point of the seminar was the papers [2] - [9] . The main subjects discussed were the refinements of uniform spaces [6] , partitions of unity on uniform spaces [6] , and a detailed study of coz-mappings and Baire mappings.

1. Cardinal reflections. Denote by p^{α} the reflection of uniform spaces on spaces having a basis for uniform covers

consisting of covers of cardinal less than $\aleph_\alpha$. Thus p^0 is the usual precompact reflection, p^1 is usually denoted by e (enumerable) in the literature. Recall that each uniform cover of X of cardinal less than $\aleph_\alpha$ is a uniform cover of $p^\alpha X$ if either the generalized continuum hypothesis is assumed ([13]) or if the σ-point-finite uniform covers of X form a basis for all uniform covers [15]. J. Pelant showed, assuming that there exists an almost disjoint family of uncountable subsets of ω_1 of cardinal 2^{ω_1} (this condition is satisfied in a model exhibited by Baumgartner) that there exists a uniform cover of $\ell_\infty(2^{\omega_1})$ of cardinal $\aleph_1$ which is not a uniform cover of $p^2 \ell_\infty(2^{\omega_1})$. The construction is combinatorially involved, and quite general, and will find further applications. J. Pelant used it to show without any set-theoretical assumption that

$\ell_\infty(2^{\omega_1})$ does not have a basis for uniform covers consisting of σ-point-finite covers.

2. The rest of the results is closely related to refinements of the category U of uniform spaces (T_2 is not assumed). Let Set^U be the category on uniform spaces (objects are uniform) having all mappings of X into Y for morphisms. Any sub-category of Set^U containing U is called a refinement of U . The definition for concrete categories as well as for general categories states obviously. It is in fact just an embedding with object function onto. Intuitively it corresponds to making the structure "less rich". Let $\mathcal{X}$ be a refinement of U , that means

$$U(X,Y) \subset \mathcal{K}(X,Y) \subset \mathrm{Set}^U(X,Y) ,$$

for each uniform space X and Y. A uniform space $X(Y)$ is called $\mathcal{K}$-coarse ($\mathcal{K}$-fine) if

$$U(X,Y) = \mathcal{K}(X,Y)$$

for each Y (X resp.). The classical refinements are

$$U \hookrightarrow \mathcal{P} \hookrightarrow C \hookrightarrow \mathrm{Set}^U ,$$

where $\mathcal{P}$ is the category of proximal maps, and C is the category of continuous maps of uniform spaces. In this case we say proximally coarse (= precompact), proximally fine, topologically ccarse (= set coarse), topologically fine (just fine by Isbell), set-fine (= uniformly discrete = the diagonal is a vicinity). Further example, cardinal refinements

$$U \hookrightarrow \mathcal{P}^{\alpha} \hookrightarrow \mathcal{P}^{0} = \mathcal{P} ;$$

the definition is:

$$\mathcal{P}^{\alpha}(X,Y) = U(Y, p^{\alpha} Y) .$$

It is obvious that the class of all $\mathcal{K}$-fine ($\mathcal{K}$-coarse) spaces is coreflective (or reflective, resp.); the corresponding coreflection is denoted by $\mathcal{K}_f$ ($\mathcal{K}_c$, resp.). A refinement $\mathcal{K}$ is called fine-maximal (coarse maximal, resp.) if

$$\mathcal{K}(X,Y) = U(\mathcal{K}_f X,Y)$$

$$(\mathcal{K}(X,Y) = U(X, \mathcal{K}_c Y), \text{ resp.}).$$

If $F\colon U \longrightarrow U$ is a coreflection, then

$$\mathcal{K}(X,Y) = U(FX, FY)$$

is a fine-maximal coreflection, and $F = \mathcal{K}_f$. Similarly, for modification-reflections. For the refinements of the category U_H of Hausdorff uniform spaces J. Vilímovský proved the following important result:

<u>Theorem</u>. If a refinement $\mathcal{K}$ of U_H is both fine-maximal and coarse maximal, then $\mathcal{K} = U_H$.

For the proof the following two results are needed:

<u>Theorem</u> (M. Hušek, J. Vilímovský). If c is a coreflection of U or U_H then either c is the identity or $pX \neq pcX$ for some X .

<u>Observation</u> (J. Vilímovský). In U_H the subcategory of all precompact spaces is the smallest modification-reflective subcategory.

<u>Remark</u>. It is an open question whether or not the theorem of Vilímovský holds for U (then, of course, $\mathcal{K} = \mathrm{Set}^U$ is the alternative).

The proof of Hušek-Vilímovský Theorem, and also the proof of the following result of M. Hušek uses the space from [1], p. 699, or J. Isbell, Uniform spaces, ex. III.3.

There is no non-trivial simultaneously reflective and coreflective subcategory of U (this property is possessed by many but not all "nice" subcategories of U).

If $\mathcal{K}$ is a refinement, and X is a space, let $\langle X \rangle_{\mathcal{K}}$ be the set of all uniform spaces Y such that the identity bijection $X \longrightarrow Y$ is an isomorphism in $\mathcal{K}$. A space X is called simply $\mathcal{K}$-fine if X is the finest element of

Obviously $\mathcal{X}$-fine is simply $\mathcal{X}$-fine. The converse holds for $\mathcal{X} = \mathcal{P}$, but it does not hold (V. Kůrková) for refinements defined by reflections on zero-dimensional spaces or on spaces with basis of star-finite covers. The validity of the converse is open for almost all p^{α} ; however, Rödl showed that the converse is false for zero-dimensional spaces and p^{1} .

The equivalence $\langle X \rangle_{\mathcal{X}}$ with morphism from $\mathcal{X}$ form a category of $\mathcal{X}$-spaces. For example, $\langle X \rangle_{\mathcal{P}}$ are proximity spaces.

3. Products of fine spaces. The attention was focused on proximity spaces. M. Hušek exhibited an example of two countable topological fine spaces such that the product is not proximally fine.

M. Hušek showed that the product of a family of spaces is proximally fine iff each finite subproduct is proximally fine. V. Kůrková and M. Hušek proved that if X and Y are proximally fine, and if Y is proximally coarse (e.g., compact), then $X \times Y$ is proximally fine.

4. Partitions of unity and distal spaces. Given a set A , let us consider the topological linear spaces R^{A} , $\ell_1(A)$ - the space of absolutely summable elements of R^{A} , $\ell_{\infty}(A)$ - the space of bounded functions on A with the sup-norm, and the subspace $c_0(A)$ of $\ell_{\infty}(A)$ consisting of all $\{x_{\alpha}\}$ such that for each $\varepsilon > 0$ the set $\{a \mid |x_a| > \varepsilon\}$ is finite. Clearly

$$\ell_1(A) \longrightarrow c_0(A) \hookrightarrow \ell_\infty(A) \longrightarrow R^A$$

Denote by $B(A)$ the unit ball in $\ell_1(A)$, and by S^+ the positive part of the unit sphere of $\ell_1(A)$. A partition of unity on X is a map $\varphi = \{\varphi_a\} : X \longrightarrow S^+(A)$. A partition of unity is called ℓ_∞, or ℓ_1, or R^A, or $\sigma(\ell_1, \ell_\infty)$ uniformly continuous if φ is uniformly continuous wrt the uniformity on S^+ inherited from the respective space. Similar notation is used for maps into $B(A)$ The topologies on S^+ inherited by the all considered uniform structures are identical. We mention three results by the author:

A space is metric-fine iff every ℓ_∞ uniformly continuous partition is ℓ_1 uniformly continuous.

A space is Alexandrov iff any ℓ_∞ uniformly continuous partition is $\sigma(\ell_1, \ell_\infty)$ uniformly continuous.

The distally coarse reflection DX of X is projectively generated by ℓ_∞ uniformly continuous partitions. (This may be improved to a Stone-Weierstrass theorem for distal spaces).

Recall [6] that a map $f: X \longrightarrow Y$ is distal if the preimages of uniformly discrete families are uniformly discrete. The distal maps form a refinement D of U which is coarse-maximal, and the distally coarse spaces are characterized by existence of a basis for uniform covers consisting of finite-dimensional covers.

M. Kosina and P. Pták gave an axiomatic characterization of the collection of all uniformly discrete families in a uniform space, and hence of the distal space (see also [16]).

There is a connection with the uniform dimension. For a partition φ let $\lambda(\varphi) = \inf \{ \| \varphi x \|_\infty \mid x \in X \}$. For a uniform cover $\mathcal{U}$ let $\lambda(\mathcal{U})$ be the supremum of $\lambda(\varphi)$ where φ runs over all ℓ_∞-uniformly continuous partitions, sub-ordinated to $\mathcal{U}$. Then the uniform dimension of X is obviously related to $\inf \{ \lambda(\mathcal{U}) \}$ (J. Hejcman).

A uniform space is called an ℓ_1-space if for every uniform cover $\mathcal{U}$ there exists an ℓ_1-uniformly continuous partition sub-ordinated to $\mathcal{U}$. The class of all ℓ_1-spaces is reflective, and M. Zahradník showed that infinite-dimensional Banach spaces are not ℓ_1-spaces.

5. <u>Coz-refinement</u>. $\mathrm{Coz}(X,Y)$ consists of all $f: X \longrightarrow Y$ such that the preimages of cozero-sets are cozero-sets. The coz-spaces (see § 2) form a subcategory of paved spaces, [3], and

$$\Pi \{ \mathrm{coz}\ X_a \} = \mathrm{coz}\ \Pi \{ pX_a \} = \mathrm{coz}\ \Pi \{ p^1 X_a \} \quad .$$

Coz-fine spaces wrt the unit interval I are called Alexandrov spaces, coz-fine classes wrt R are called inversion-closed spaces. For a long list of characteristic properties see [10]. We add a result of D. Preiss with a nice proof by M. Zahradník that inversion-closed spaces have the Daniel property, i.e. each sequence $\{ f_n \}$ of bounded uniformly continuous functions decreasing to 0 is equi-uniformly continuous.

Hereditarily Alexandrov spaces X (equivalently, bounded coz-functions extend from subspaces) are characterized by each of the following conditions (the author):

a) The Samual compactification is hereditarily Alexandrov.

b) X is Alexandrov, and the cozero sets form a normal paving.

c) X is Alexandrov, and if $Y \subset X$, and Z_1 and Z_2 are zero-sets on X such that $Y \cap Z_1 \cap Z_2 = \emptyset$, then there exist disjoint zero-sets Z_1' and Z_2' such that $Z_i' \supset Z_i \cap Y$.

Hereditarily inversion-closed spaces are just the spaces with the property that the countable partitions by Baire sets are uniform covers ([8]). Consequently, for refinements $\mathcal{H}$ larger than that given by inversion-closed spaces we get that for hereditarily $\mathcal{H}$-fine the Baire sets are just cozero sets, and hence, e.g., hereditarily coz-fine is Baire-fine. On the other hand, the identification of hereditarily coz-fine spaces among Baire-fine spaces seems to be a difficult open problem. The measurable coreflection of a complete metric space is actually Baire-fine because the defect in the proof [4] was overcome by a simple (however, deep) lemma by D. Preis [14].

7. The locally e-fine metric-fine spaces [9] are characterized as follows (the author):

if $f: X \longrightarrow S$, S metric, is uniformly continuous, and

if $G = \check{f}^{-1}[S]$ where $\check{f}: \check{X} \longrightarrow \check{S}$ is the extension to the Samual compactifications, then the identity of X into G with the topological fine uniformity is uniformly continuous.

This approach applies to metric-fine replaced by: metric-λ = sub(metric-fine) = (complete metric)-fine.

8. Atoms. An atom is a uniform space X which is not set-fine (= uniformly discrete), and there is no uniform structure between that of X and the set-fine uniformity on X. J. Pelant and J. Reiterman proved:

a. If X is T_2, then X is an atom and pX is not set-fine proximity iff there exist disjoint sets A and B of X, and a bijection $\varphi : A \longrightarrow B$, and an ultrafilter $\mathcal{A}$ on A such that the covers $\mathcal{U}(\alpha)$, $\alpha \in \mathcal{A}$, form a basis:

$\mathcal{U}(\alpha)$ is the collection of all two-point sets $\{a, \varphi a\}$, $a \in \alpha$, and all singletons (x), $a \neq x \neq \varphi a$ for a in α.

b. If $\mathcal{A}$ is an ultrafilter on X then the uniformity which has all covers $\mathcal{V}(\alpha)$, $\alpha \in \mathcal{A}$, for a basis, where $\mathcal{V}(\alpha)$ consists of α and all (x), $x \in X - \alpha$, is an atom iff $\mathcal{A}$ is selective.

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