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EXISTENCE THEOREMS FOR A VARIANT OF HAMMERSTEIN'S INTEGRAL EQUATION

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Abstract: Existence theorems are obtained for a variant of Hammerstein's integral equation of the type $u(s) + \int_\Omega k(s,t)\, f(t,u(t), Bu(t))dt = 0$ where B is a bounded linear operator from a closed subspace of L^p to $L^q (\frac{1}{p} + \frac{1}{q} = 1)$. The kernel K is assumed to be such that the linear integral operator A given by $Au(s) = \int_\Omega K(s,t)\, u(t)dt$ is compact and angle-bounded. The function f satisfies the usual Nemytskii type conditions and the condition $uf(t,u,v) \geq c|u|^r |v|^s$, $\frac{r}{p} + \frac{s}{q} = 1$ for sufficiently large u and all v.

Key words: Hammerstein equation, angle-bounded operator, Caratheodory conditions.

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1. Introduction. A nonlinear integral equation of Hammerstein type is of the form

$$u(s) + \int_\Omega K(s, t)f(t, u(t))dt = 0 . \tag{1}$$

Usually one assumes that Ω is a measurable subset of R^n, $f(t,u)$ is a function of the variables $t \in \Omega$, $u \in R$ satisfying the so-called Carathéodory conditions i.e. $f(t,u)$ is continuous with respect to u for almost all $t \in \Omega$ and measurable with respect to t for all values of u. There is an extensive literature on Hammerstein equations with contributions by Hammerstein [7], Iglish [8], Golomb [6], Dolph [4], Rothe [17], Vainberg [18], Krasnoselskii [13] and others. In recent years monotonicity concepts have lead to the detailed study of a more abstract Hammerstein type equation by many authors which include Kachurovsky [9], Vainberg [18], Dolph-Minty [5], Kolodner [10], Brézis [2], Kolomý [11], Amann [1] and Browder-Gupta [3]. The abstract form of Hammerstein's equation is

$$u + KNu = 0 \tag{2}$$

where K is a linear mapping and N a nonlinear mapping. In the case of equation (2) the corresponding mappings are given by

$$Kv(s) = \int_\Omega K(s, t)v(t)dt, \quad Nu(s) = f(s, u(s)) . \tag{3}$$

In this paper we obtain existence theorems in a closed subspace of $L^p = L^p(\Omega)$ for the following variant of Hammerstein's integral equation

$$u(s) + \int_\Omega K(s, t)f(t, u(t), Bu(t))dt = 0 . \tag{4}$$

Here f is a function which satisfies Carathéodory conditions as a function of three variables, B is a linear bounded map from a closed subspace Y of L^p to L^q.

We define the Nemytskii operator G on a space of pair of functions by

$$G(u, v)(s) = f(s, u(s), v(s)) . \tag{5}$$

The following lemma is proved as the corresponding one Krasnoselskii [13].

Lemma 1. Suppose that the operator G maps all of $L^p \times L^q$ into L^q, where $\frac{1}{p} + \frac{1}{q} = 1$, $p > 1$. Then the operator G is continuous and bounded.

We now define a new operator F on the space Y by $Fu = G(u, Bu)$, or

$$\text{(6)} \qquad Fu(t) = f(t, u(t), Bu(t)), \; u \in Y$$

and a linear integral operator A on L^p by

$$\text{(7)} \qquad Au(s) = \int_\Omega K(s, t)u(t) .$$

We have the following lemmas.

Lemma 2. Let the function f be such that the operator G given by (5) maps all of $L^p \times L^q$ into L^q. Then the operator F given by (6) is a continuous bounded map from Y to L^q, $(p > 1)$.

Proof: Let $G(u,v)(t) = f(t,u(t),v(t))$ and $ju = \{U, Bu\}$, then $F = G \circ j$. Since G maps $L^p \times L^q$ to L^q, by Lemma 1 G is a continuous and bounded map from $L^p \times L^q$ to L^q. Since j is a continuous map from Y to $L^p \times L^q$, it follows that the composite map $G \circ j = F$ is a continuous and bounded map from Y to L^q.

Definition 1. If X is a real Banach space and X^* its dual, we denote by $\langle w,u \rangle$ the duality pairing between the element w of X^* and the element u of X. A mapping A of X into X^* is said to be monotone if for all u, v in

X we have

(8) $$\langle Au - Av,\ u - v \rangle \geq 0 .$$

We now define angle-bounded map, for reference see Browder and Gupta [3].

<u>Definition 2</u>. If A is a bounded monotone linear map of X into X^*, then A is said to be angle-bounded with constant $\alpha \geq 0$ if for all u, v in X we have

(9) $$|\langle Au, v \rangle - \langle Av, u \rangle| < 2\alpha \{\langle Au, u \rangle\}^{1/2} \{\langle Av, v \rangle\}^{1/2} .$$

It is clear that every monotone map A which is symmetric (i.e. $\langle Au, v \rangle = \langle Av, u \rangle$ for all u, v in X) is angle-bounded with $\alpha = 0$.

Hereafter we shall make use of the following theorems of Amann [1] for the abstract equation of Hammerstein type (2).

<u>Theorem 1</u> (Amann). Let X be an arbitrary Banach space and let $A: X \longrightarrow X^*$ be a linear, injective, monotone compact operator. Let Y be a closed subspace of X^* which contains the range of A. Let $F: Y \longrightarrow X$ be continuous and bounded and assume that there exists a constant $\rho_0 > 0$ such that

(10) $$\langle u, A^{-1}u \rangle + \langle u, Fu \rangle > 0 \quad \text{for} \quad u \in R(A) \ \text{and} \ \|u\| > \rho_0 .$$

Then the Hammerstein operator equation

(11) $$u + AFu = 0$$

has at least one solution u in Y. Moreover every solution satisfies $\|u\| \leq \rho_0$.

Theorem 2. Let X be an arbitrary Banach space and let $A: X \to X^*$ be linear, angle-bounded with constant $\alpha \geq 0$ and compact. Let Y be a closed subspace of X^* which contains the range of A. Let $F: Y \to X$ be continuous and bounded and assume that there exists a number $\rho_0 > 0$ such that for all $u \in R(A)$

$$(12) \qquad \langle u, Fu \rangle \geq -(1 + \alpha^2)^{-1} \| A \|^{-1} \| u \|^2$$

for all $\| u \| > \rho_0$.

Then the Hammerstein equation (11) has at least one solution u in Y for which $\| u \| \leq \rho_0$.

2. Existence theorems. In the following theorems $p > 1$, and $|\Omega| < \infty$.

Theorem 3. Suppose

(i) the kernel K is such that the linear integral operator A defined by (7) is compact monotone and its range is contained in Y which is a closed subspace of L^p.

(ii) B is a linear bounded operator from Y to L^q and also from L^∞ to L^∞. Further it satisfies the condition

$$(13) \qquad \int_\Omega Bu(t)u(t)dt \geq 0 \quad \text{for all } u \text{ in } Y.$$

(iii) The function f is such that the operator G given by (5) maps all of $L^p \times L^q$ to L^q. Also assume that $\sup_{|u| \leq \sigma, |v| \leq b\sigma} |f(t,u,v)|$ is in $L^1(\Omega)$ where $\sigma > 0$ is such that

(14) $uf(t,u,v) \geq c\,|u|^p + duv$ for $|u| > \sigma$, $v \in \mathbb{R}$,

$c > 0$, $d \geq 0$.

Then the integral equation

$$(*)\qquad u(s) + \int_\Omega K(s,t)f(t,u(t),Bu(t))dt = 0$$

has a solution u in Y such that $\|u\| \leq \rho_0$, where ρ_0 is such that

$$(15)\qquad \rho_0^{n} = \frac{1}{c}\left[c\sigma^p\,|\Omega| + a(\sigma) + db\sigma^2|\Omega|\right].$$

Here $a(\sigma)$ denotes the L^1 norm of $\sup_{|u|\leq\sigma,\,|v|\leq b\sigma}|f(t,u,v)|$, b the L^∞ to L^∞ operator norm of B and $\|u\|$ the L^p norm of u.

<u>Proof</u>. The assertion will follows from Theorem 1. We set $X = L^q$ and define F and A as in (6) and (7). Then $X^* = L^p$ and $(*)$ is equivalent to the operator equation

$$(***)\qquad u + AFu = 0.$$

Since F satisfies all the conditions of Lemma 2 it follows that F is a continuous bounded mapping from Y to X. Similarly A is a continuous, monotone and compact map from Y to X^* whose range is contained in Y. Furthermore by (13) and (14) we claim that $\langle u,Fu\rangle > 0$ for $\|u\| > \rho_0$ where

$$\langle u,Fu\rangle = \int_\Omega u(t)f(t,u(t),Bu(t))dt.$$

Assume to the contrary that

$$\int_\Omega u(t)f(t,u(t),Bu(t))dt \le 0 ,$$

for some u , $\|u\| > \rho_0$. Then

$$\int_\Omega |u|^p = \int_{M=\{t:|u(t)|\le\sigma\}} |u|^p + \int_{M^c}|u|^p \le \sigma^p|\Omega| + \int_{M^c}|u|^p$$

$$\le \sigma^p|\Omega| + \frac{1}{c}\int_{M^c}[u(t)f(t,u(t),Bu(t)) -$$

$$- du(t)Bu(t)]\, dt = \sigma^p|\Omega| + \frac{1}{c}\int_\Omega u(t)f(t,u(t),$$

$$Bu(t))dt - \frac{d}{c}\int_\Omega u(t)Bu(t)dt - \frac{1}{c}\int_M u(t)f(t,u(t),$$

$$Bu(t))dt + \frac{d}{c}\int_M u(t)Bu(t)dt \le \sigma^p|\Omega| +$$

$$+ \frac{1}{c}\int_M |u(t)|\,|f(t,u(t),Bu(t))|\, dt +$$

$$+ \frac{d}{c}\int_M |u(t)|\,|Bu(t)|\, dt \le \sigma^p|\Omega| +$$

$$+ \frac{\sigma}{c}\int_M \sup_{|u|\le\sigma,\,|v|\le b\sigma} |f(t,u,v)|\, dt +$$

$$+ \frac{d\sigma}{c}\int_M |Bu(t)|dt \le \sigma^p|\Omega| + \frac{\sigma}{c}a(\sigma) + \frac{d\sigma^2 b}{c}|\Omega|$$

$$= \frac{1}{c}[c\sigma^p|\Omega| + \sigma a(\sigma) + db\sigma^2|\Omega|]$$

i.e. $\|u\| \le \rho_0$, a contradiction.

Thus F and A in the operator equation (***) satisfy all the conditions of Theorem 1 and therefore the result follows.

If the operator A is assumed to be angle-bounded, then the hypothesis on the operator B can be relaxed as we see in the following theorem.

__Theorem 4.__ Suppose

(i) the kernel K is such that the linear integral operator A defined by (7) is compact, angle-bounded with constant $\alpha \geq 0$ and its range is contained in Y, a closed subspace of L^p.

(ii) B is a linear bounded operator from Y to L^q and also from L^∞ to L^∞.

(iii) The function f is such that the operator G given by (5) maps all of $L^p \times L^q$ to L^q. Also $\sup_{|u| \leq \sigma, |v| \leq b\sigma} |f(t,u,v)|$ is in $L^1(\Omega)$, where $\sigma > 0$ is such that

$$(16) \qquad uf(t,u,v) \geq -c\,|u|^r\,|v|^s \quad \text{for } |u| > \sigma,\ v \in \mathbb{R}$$

$$\frac{r}{p} + \frac{s}{q} = 1,\ r + s \leq 2.$$

Then if

$$(17) \qquad \sigma a(\sigma)\rho_0^{-2} + c\,\|B\|^s\,\rho_0^{\,r+s-2} < (1 + \alpha^2)^{-1}\,\|A\|^{-1},$$

the integral equation $(*)$ has a solution u in Y satisfying $\|u\| \leq \rho_0$. Here $a(\sigma)$, b and $\|u\|$ are as defined in Theorem 3, $\|B\|$ the $L^p \longrightarrow L^q$ operator norm of B.

__Proof.__ The assertion will follow from Theorem 2. As before we set $X = L^q$ and define the operators F and A as in (6) and (7) respectively. Then $X^* = L^p$ and $(*)$ is equivalent to the operator equation

$$(***) \qquad u + AFu = 0$$

F is a continuous bounded map from Y to X. By hypothesis on the kernel K , A is a continuous, angle bounded, compact map from X to X^* whose range is contained in Y . Furthermore by (16) we have

$$\int_\Omega u(t)Fu(t)dt = \int_\Omega u(t)f(t,u(t), B\,u(t))\,dt$$

$$= \int_{\{t:|u(t)|>\sigma\}} u(t)f(t,u(t), Bu(t))\,dt +$$

$$+ \int_{M=\{t:|u(t)|\le\sigma\}} u(t)f(t,u(t), Bu(t))\,dt$$

$$\ge -c\int_\Omega |u(t)|^r\,|Bu(t)|^s\,dt - \int_M |u|\,|f(t,u(t), Bu(t))|\,dt$$

$$\ge -c\left(\int_\Omega |u|^p\right)^{r/p}\left(\int_\Omega |Bu|^q\right)^{s/q}$$

$$-\sigma\int_\Omega \sup_{|u|\le\sigma,\,|v|<b\sigma} |f(t,u,v)|\,dt$$

$$= -c\,\|u\|^r\,\|Bu\|_q^s - \sigma a(\sigma) \ge -c\,\|B\|^s\,\|u\|^{r+s} - \sigma a(\sigma)\,.$$

Using (17) we have

$$\langle u, Fu\rangle \ge -(1+\alpha^2)^{-1}\ \|A\|^{-1}|\rho_0^{-2} \quad \text{for } \|u\|>\rho_0\,.$$

Thus

$$\langle u, Fu\rangle \ge -(1+\alpha^2)^{-1}\,\|A\|^{-1}\,\|u\|^2 \quad \text{for } \|u\|>\rho_0\,.$$

Since the operators A and F satisfy all the hypotheses of Theorem 2 (***) has a solution u in Y such that $\|u\| \le \rho_0$. This implies that (*) has a solution u in $L^p(\Omega)$ satisfying $\|u\| \le \rho_0$.

Remark. (17) is satisfied for all sufficiently large ρ_0 if either $r + s < 2$ or $r + s = 2$ and $c \| B \|^{s} < (1 + \alpha^2)^{-1} \| A \|^{-1}$. In these two cases $(*)$ has a solution in $L^p(\Omega)$.

If f does not depend on v, we obtain the following existence theorem for Hammerstein equation

$$(18) \qquad u(s) + \int_{\Omega} K(s,t) f(t, u(t))dt = 0$$

as a corollary to Theorem 4.

Corollary 1. Suppose

(i) the kernel $K(s,t)$ satisfies condition (i) of Theorem 4.

(ii) The function f is such that the operator F maps L^p to L^q and for some $\sigma > 0$ and $\sup_{|u| \leq \sigma} |f(t,u)|$ is in L^1 and

$$(19) \qquad uf(t,u) \geq -c|u|^p \quad \text{for} \quad |u| > \sigma .$$

If

$$(20) \qquad \sigma a(\sigma) \rho_0^{-2} + c \rho_0^{p-2} < (1 + \alpha^2)^{-1} \| A \|^{-1}, \; p \leq 2 ,$$

then the Hammerstein equation (18) has a solution u in L^p with $\| u \| \leq \rho_0$.

Proof. This is a direct consequence of Theorem 4.

If the operator B is defined on the whole space L^p, in particular by the kernel K_1 as

$$(21) \qquad Bu(s) = \int_{\Omega} K_1(s,t) u(t) dt$$

then we obtain the following existence theorem for the integral equation

(22) $$u(s) + \int_\Omega K(s,t) f(t\ u(t), \int_\Omega K_1(s,\tau) u(\tau) d\tau) dt = 0$$

as a corollary to Theorem 4.

Corollary 2. Suppose

(i) the kernel K satisfies condition (i) of Theorem 4.

(ii) The kernel K_1 is such that the operator B is a bounded operator from L^p to L^q and also from L^∞ to L^∞.

(iii) The function f satisfies condition (iii) of Theorem 4.

Then the integral equation (22) has a solution u in L^p with $\| u \| \leq \rho_0$, where ρ_0 is a positive number satisfying (17).

Proof. This is a direct consequence of Theorem 4.

Remark. Existence and uniqueness of the solution of integral equation (22) have been discussed by Nesterenko [16], who uses the method of degenerate kernels.

3. Nonnegative solutions

Definition 3. Let X be a Banach space. A set $K \subseteq X$ is called a cone if the following conditions are satisfied:

(a) the set K is closed,

(b) if $u, v \in K$ then $\alpha u + \beta v \in K$ for all $\alpha, \beta \geq 0$,

(c) for $u \neq 0$, $u \in K$, there is $-u \notin K$.

Nonnegative functions form a cone in L^p spaces. Existence of nonnegative solutions of the operator equations has been discussed in detail by Krasnoselskii [14] with applications to non-linear integral equations and boundary value problems. In this section we shall discuss about the existence of the operator equation

$$u = AFu \tag{23}$$

in a cone. Here A and F are operators as defined earlier. The operators A and F are assumed to be such that A maps a cone K_2 into a cone K_1 and F maps K_1 into K_2. We have the following theorem as an easy generalization of Theorem 2 for the operator equation (23).

<u>Theorem 5</u>. Suppose X is a real Banach space X^* its dual and $A: X \longrightarrow X^*$ is linear, angle-bounded with constant $\alpha \geq 0$ and compact and its range is contained in a closed subspace Y of X^*. Further assume that $A(K_2) \subseteq K_1$ where K_2 is a cone in X and K_1 is a cone in Y. Let $F:$ $: K_1 \longrightarrow K_2$ be continuous and bounded and assume that there exists a constant $\rho_0 > 0$ such that

$$\langle u, Fu \rangle < (1 + \alpha^2)^{-1} \|A\|^{-1} \|u\|^2 \quad \text{for all } u \in K_1 \text{ and } \|u\| > \rho_0 . \tag{24}$$

Then the operator equation (23) has a solution u in K_1 with $\|u\| \leq \rho_0$.

As a consequence of the above theorem, we obtain the following theorems for non-linear Hammerstein type integral equations. It is interesting to note that as a corollary we obtain results similar to those of Krasnoselskii [14] and Hammerstein [7].

Theorem 6. Suppose

(i) the kernel K is such that the operator A defined by it is angle-bounded (with constant $\alpha \geq 0$) and compact operator from L^q to L^p $(1 < p \leq 2 , \frac{1}{p} + \frac{1}{q} = 1)$; moreover $K(s,t) \geq 0$ for all $s,t \in \Omega$,

(ii) the function f satisfies the Carathéodory conditions and

$$0 \leq f(t,u) \leq a(t) + bu^r , \quad u \geq 0 \tag{25}$$

$$a \in L^q , \quad b > 0 \quad r \leq p - 1 .$$

If ρ_0 is a positive number such that

$$\rho_0^{-1} \| a \| + \rho_0^{r-1} b |\Omega|^{(1-\frac{r+1}{p})} < (1 + \alpha^2)^{-1} \| A \|^{-1} \tag{26}$$

then the integral equation

$$u(s) = \int_\Omega K(s,t) f(t,u(t)) dt \tag{27}$$

has a nonnegative solution u in L^p satisfying $\| u \| \leq \rho_0$.

Proof. We take K_1 and K_2 as cones of nonnegative function and then proceed as in Theorem 4.

Remark 3. (26) is satisfied for all sufficiently large ρ_0 if either $r < 1$, or $r = 1$ and $b |\Omega|^{(1-2/p)} < (1 + \alpha^2)^{-1} \| A \|^{-1}$. In these two cases (27) always has a non-negative solution in L^p . In view of Remark 3, we obtain the following corollary.

Corollary 3. Suppose

(i) the kernel K is such that the operator A is angle-bounded (with constant $\alpha \geq 0$) compact operator from L^2 to L^2 , and $K(s,t) \geq 0$ for $s, t \in \Omega$.

(ii) The function f satisfies the Carathéodory conditions and

$$(28) \qquad 0 \leq f(t,u) \leq a(t) + bu, \quad u \geq 0$$
$$a \in L^2, \quad b > 0$$

$$(29) \qquad b(1 + \alpha^2) \|A\| < 1 .$$

Then the integral equation (27) has a nonnegative solution u in L^2 .

For $\alpha = 0$ (symmetric kernel) this reduces to one of Hammerstein's original results [7].

We now give a similar theorem for the integral equation

$$u(s) = \int_\Omega K(s,t) f(t,u(t), Bu(t))dt$$

Theorem 7. Suppose

(i) the kernel K satisfies all the conditions of Theorem 5 with the additional hypothesis that the range of A is contained in a closed subspace Y of L^p .

(ii) B is a bounded linear operator from Y to L^q .

(iii) The function f satisfies Carathéodory conditions and

(30) $$0 \le f(t,u,v) \le a(t) + b_1 u^r + b^2 u^\beta |v|^r , \quad u \ge 0 , \quad v \in R$$

$$a \in L^q , \ b_1 > 0 , \ b_2 > 0 , \ r \le p - 1 ,$$

$$\beta + \gamma \le p - 1, \quad \frac{\beta+1}{p} + \frac{\gamma}{q} = 1 .$$

If ρ_0 is a positive number such that

(31) $$\rho_0^{-1} \|a\| + \rho_0^{r-1} b_1 |\Omega|^{(1-\frac{r+1}{p})} + \rho_0^{\beta+\gamma-1} b_2 \|B\|^\gamma < (1+\alpha^2)^{-1} \|A\|^{-1} .$$

Then the integral equation

(32) $$u(s) = \int_\Omega K(s,t) f(t, u(t), Bu(t)) dt$$

has a nonnegative solution u satisfying $\|u\| \le \rho_0$.

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