Bohdan Zelinka
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TOLERANCE RELATIONS ON SEMILATTICES

Bohdan ZELINKA, Liberec

Abstract: A tolerance compatible with an algebra is defined similarly as a congruence, only the transitivity is not required. This paper contains some results on tolerances compatible with a semilattice.

Key words: Semilattice, tolerance.

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This paper continues the study of tolerance relations on algebras which was begun in [2],[3] and [4]. The concept of tolerance was introduced by E.C. Zeeman [1].

A tolerance relation is a binary relation on some set which is reflexive and symmetric. If \( \mathcal{U} = \langle A, \mathcal{F} \rangle \) is some algebra (\( A \) denotes the set of elements of \( \mathcal{U} \) and \( \mathcal{F} \) denotes the set of its operations), and \( \xi \) is some tolerance on \( A \), we say that \( \xi \) is compatible with \( \mathcal{U} \) if and only if the following condition is satisfied: If \( f \in \mathcal{F} \) is an \( n \)-ary operation, where \( n \) is a positive integer, and \( x_1, \ldots, x_n \), \( y_1, \ldots, y_n \) are elements of \( A \) such that \((x_i, y_i) \in \xi\) for \( i = 1, \ldots, n \), then

\[
(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n)) \in \xi.
\]

Here we shall study tolerances on semilattices. If a semilattice is not considered as a part of a lattice, the
operation in it is called multiplication and denoted by \( \circ \), its result is called product. The ordering on a semilattice \( S \) is defined so that for \( a \in S \), \( b \in S \) we have \( a \preceq b \) if and only if \( a \circ b = b \). If a semilattice is considered as a part of a lattice, we use the signs \( \lor \) and \( \land \) for the lattice operations and call them join and meet.

Thus, if \( S \) is a semilattice and \( \xi \) is a tolerance on the set of elements of \( S \), then \( \xi \) is compatible with \( S \) if and only if for any \( x_1 \in S \), \( x_2 \in S \), \( y_1 \in S \), \( y_2 \in S \) such that \( (x_1, y_1) \in \xi \), \( (x_2, y_2) \in \xi \) we have \( (x_1 \circ x_2, y_1 \circ y_2) \in \xi \).

**Theorem 1.** Let \( S \) be a semilattice, let \( \xi \) be a tolerance compatible with \( S \). Let \( x \in S \). The set \( S(x) = \{ y \in S \mid (x, y) \in \xi \} \) is a subsemilattice of \( S \). Moreover, if \( S(x) \) has the greatest element \( M(x) \) for each \( x \in S \), then the mapping \( M \) which assigns \( M(x) \) to \( x \) for each \( x \in S \) is an isotone mapping of \( S \) into itself.

**Proof.** A semilattice is a commutative semigroup in which all elements are idempotents. Thus \( \{ x \} \) for each \( x \in S \) is a subsemilattice of \( S \) and according to Theorem 4 from [2] also \( S(x) \) is a subsemilattice of \( S \). The assertion for \( M(x) \) is proved analogously to the proof of Theorem 12 from [2]; that theorem is proved for lattices, but in its proof no meets are used.

Now if \( a \in S \), \( b \in S \), \( a \preceq b \), then the interval \( \langle a, b \rangle \) is by definition the set \( \{ x \in S \mid a \preceq x \preceq b \} \).

**Theorem 2.** Let \( S \) be a semilattice, let \( \xi \) be a to-
lerance compatible with $S$. Let $x \in S$, $y \in S$, $(x,y) \in \xi$. Then $(x \circ y,z) \in \xi$ for each $z \in \langle x, x \circ y \rangle \cup \langle y, x \circ y \rangle$.

**Proof.** Let $z \in \langle x, x \circ y \rangle \cup \langle y, x \circ y \rangle$. We have $(x,y) \in \xi$, $(z,z) \in \xi$, therefore $(x \circ z, y \circ z) \in \xi$. Evidently, $\langle x, x \circ y \rangle \cup \langle y, x \circ y \rangle = \langle x \circ y \rangle$, thus $y \circ z = x \circ y$ for each $z \in \langle x, x \circ y \rangle$ and $x \circ z = x \circ y$ for each $z \in \langle y, x \circ y \rangle$. Thus if $z \in \langle x, x \circ y \rangle$, we have $z \geq x$, thus $x \circ z = z$ and further $y \circ z = x \circ y$; this means $\xi \ni (x \circ z, y \circ z) = (z, x \circ y)$. If $z \in \langle y, x \circ y \rangle$ then $x \circ z = x \circ y$, $y \circ z = z$ and we have again $(z, x \circ y) \in \xi$.

This is a substantial difference in comparison with the case of lattices [4]. In the case of semilattices it is not necessary that any two elements of $\langle x, x \circ y \rangle \cup \langle y, x \circ y \rangle$ should be in $\xi$. For example, let $C_1$, $C_2$ be two disjoint chains of the cardinality greater than one with the least elements $c_1$, $c_2$ respectively, let $0$ be an element which does not belong to $C_1 \cup C_2$. Put $S = C_1 \cup C_2 \cup \{0\}$ and define the ordering in $S$ so that $x \leq y$ if and only if either both $x$ and $y$ are in $C_1$ and $x \leq y$ holds in $C_1$, or both $x$ and $y$ are in $C_2$ and $x \leq y$ holds in $C_2$, or $y = 0$ and $x$ is an arbitrary element of $S$. Let $\xi$ be a tolerance relation on $S$ consisting of the pairs $(c_1,c_2)$, $(c_2,c_1)$ and the pairs $(x,x)$, $(x,0)$, $(0,x)$ for each $x \in S$. The tolerance $\xi$ is compatible with $S$.

**Theorem 3.** Let $S$ be a semilattice with more than two elements. Then there exists a tolerance $\xi$ compatible with
Proof. At first let \( S \) be a chain. Let \( a \) be an element of \( S \) which is neither the greatest, nor the least one; such an element exists, because \( S \) has at least three elements. Let \( \xi \) consist of all pairs \((x, y)\), where either simultaneously \( x \sqsupseteq a \), \( y \sqsupseteq a \), or simultaneously \( x \sqsubseteq a \), \( y \sqsubseteq a \). Let \((x_1, y_1) \in \xi\), \((x_2, y_2) \in \xi\). If at least one of these pairs has the property that both elements are greater than or equal to \( a \), then \( x_1 \circ x_2 \sqsupseteq a \), \( y_1 \circ y_2 \sqsupseteq a \) and \((x_1 \circ x_2, y_1 \circ y_2) \in \xi\). If \( x_1 \sqsubset a \), \( x_2 \sqsubseteq a \), \( y_1 \sqsubseteq a \), \( y_2 \sqsubseteq a \), then also \( x_1 \circ x_2 \sqsubseteq a \), \( y_1 \circ y_2 \sqsubseteq a \) and again \((x_1 \circ x_2, y_1 \circ y_2) \in \xi\). Thus \( \xi \) is compatible with \( S \). Now let \( b < a < c \). We have \((b, a) \in \xi\), \((a, c) \in \xi\), but \((b, c) \notin \xi\), thus \( \xi \) is not transitive and it is not a congruence.

Now suppose that \( S \) is not a chain. Let \( a, b \) be two incomparable elements of \( S \). Take a tolerance \( \xi \) consisting of the pairs \((x, x)\), \((y, a \circ b)\), \((a \circ b, y)\), \((y \circ x, a \circ b \circ x)\), \((a \circ b \circ x, y \circ x)\) for each \( x \in S \), \( y \in \langle a, a \circ b \rangle \cup \langle b, a \circ b \rangle \). This is evidently a tolerance compatible with \( S \). We have \((a, a \circ b) \in \xi\), \((a \circ b, b) \in \xi\); but \((a, b) \notin \xi\), because \( a \neq b \) and none of the elements \( a, b \) can be equal to \( a \circ b \) or \( a \circ b \circ x \) for some \( x \in S \).

Now we shall consider upper and lower semilattices of a lattice.

Theorem 4. Let \( L \) be a lattice with more than two elements, let \( L(\lor) \) be the upper semilattice of \( L \), let
Let $\textnormal{L}(\land)$ be the lower semilattice of $\textnormal{L}$. Then there exist tolerances $\xi$, $\xi'$ on $\textnormal{L}$ such that $\xi$ is compatible with $\textnormal{L}(\lor)$, $\xi'$ is compatible with $\textnormal{L}(\land)$, but none of them is compatible with $\textnormal{L}$.

**Proof.** Suppose that $\textnormal{L}$ is not a chain. Then there exist elements $a$, $b$ of $\textnormal{L}$ which are incomparable. We construct the tolerance $\xi$ analogously as in the proof of Theorem 3; the tolerance $\xi$ is compatible with $\textnormal{L}(\lor)$.

Suppose that it is compatible with $\textnormal{L}$. From $(a, a \lor b) \in \xi$, $(a \lor b, b) \in \xi$ we obtain $(a \land (a \lor b), (a \lor b) \land b) = (a, b) \in \xi$, which is a contradiction. If $\textnormal{L}$ is a chain, let $a$ be an element of $\textnormal{L}$ to which at least two elements $b$, $c$ exist such that $b < c < a$. Then $\xi$ consists of the pairs $(x, x)$, $(a, y)$, $(y, a)$ for all $x \in \textnormal{S}$ and all $y \not\in a$.

Let $x_1$, $x_2$, $y_1$, $y_2$ be elements of $\textnormal{L}$, $(x_1, y_1) \in \xi$, $(x_2, y_2) \in \xi$. If $x_1 = y_1$, $x_2 = y_2$, then $x_1 \lor x_2 = y_1 \lor y_2$ and $(x_1 \lor x_2, y_1 \lor y_2) \in \xi$. If $x_1 = a$, $y_1 \not\in a$, $x_2 = y_2 = a$, then $x_1 \lor x_2 = x_2$, $y_1 \lor y_2 = y_2$ and $(x_1 \lor x_2, y_1 \lor y_2) = (x_2, y_2) \in \xi$.

If $x_1 = a$, $y_1 \not\in a$, $x_2 = y_2 \not\in a$, then $x_1 \lor x_2 = a$, $y_1 \lor y_2 \not\in a$, thus again $(x_1 \lor x_2, y_1 \lor y_2) \in \xi$. If $x_1 = a$, $y_1 \not\in a$, $x_2 = a$, $y_2 \not\in a$ or $x_1 = a$, $y_1 \not\in a$, $x_2 \not\in a$, $y_2 = a$, then $x_1 \lor x_2 = a$, $y_1 \lor y_2 \not\in a$, and $(x_1 \lor x_2, y_1 \lor y_2) \in \xi$. All other cases are obtained from some of these cases by changing the notation. Thus $\xi$ is compatible with $\textnormal{L}(\lor)$. Now let $c < d < a$. We have $(c, a) \in \xi$, $(a, d) \in \xi$, but $(c, d) = (c \land a, a \land d) \notin \xi$; the tolerance $\xi$ is not compatible with $\textnormal{L}$. The construction of $\xi'$ is dual to this.
construction.

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Československo

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