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COMMENTATIONES MATHEMATICAE UNIVERSITATIS CAROLINAE

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# ON MINIMAL SURFACES IN $E^5$

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Abstract: A global characterization of minimal surfaces of $E^5$ which are situated in $E^4$.

Key words: Minimal surface, generalized analytic function.

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We are going to prove the following

Theorem. Let $D \subset \mathbb{R}^2$ be a bounded domain and $M: D \to E^5$ a minimal surface such that $\dim T_m^2(M) = 4$ for each point $m$ of the surface $M \equiv M(D)$, $T_m^2(M)$ being the 2-osculating space of $M$ at $m$. Let $n_m \perp T_m^2(M)$ be the unit normal vector at $m$ and $S = \{m \in M ; (dn)_m = 0\}$. Then $S$ consists of isolated points or $M$ is situated in $E^4 \subset E^5$.

Proof. To each point $m \in M$, associate an orthonormal frame $\{m, v_1, v_2, v_3, v_4, v_5\}$ such that $T_m(M) = \{v_1, v_2\}$, $T_m^2(M) = \{v_1, v_2, v_3, v_4\}$, $n_m = v_5$. Then

$$(1) \qquad \begin{aligned} dm &= \omega^1 v_1 + \omega^2 v_2 , \\ dv_1 &= \omega_1^2 v_2 + \omega_1^3 v_3 + \omega_1^4 v_4 , \\ dv_2 &= -\omega_1^2 v_1 + \omega_2^3 v_3 + \omega_2^4 v_4 , \\ dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 + \omega_3^4 v_4 + \omega_3^5 v_5 , \end{aligned}$$

$$dv_4 = -\omega_1^4 v_1 - \omega_2^4 v_2 - \omega_3^4 v_3 + \omega_4^5 v_5 ,$$
$$dv_5 = - \omega_3^5 v_3 - \omega_4^5 v_4$$

with the well known integrability conditions. From

$$\omega^3 = \omega^4 = \omega^5 = 0 , \quad \omega_1^5 = \omega_2^5 = 0 , \tag{2}$$

we get

$$\begin{aligned} &\omega^1 \wedge \omega_1^3 + \omega^2 \wedge \omega_2^3 = 0 , \quad \omega^1 \wedge \omega_1^4 + \omega^2 \wedge \omega_2^4 = 0 , \\ &\omega_1^3 \wedge \omega_3^5 + \omega_1^4 \wedge \omega_4^5 = 0 , \quad \omega_2^3 \wedge \omega_3^5 + \omega_2^4 \wedge \omega_4^5 = 0 \end{aligned} \tag{3}$$

and the existence of functions $a_i$, $b_i$, $c_\alpha$ such that

$$\begin{aligned} &\omega_1^3 = a_1 \omega^1 + a_2 \omega^2 , \quad \omega_1^4 = b_1 \omega^1 + b_2 \omega^2 , \\ &\omega_2^3 = a_2 \omega^1 + a_3 \omega^2 , \quad \omega_2^4 = b_2 \omega^1 + b_3 \omega^2 , \end{aligned} \tag{4}$$

$$\begin{aligned} &a_1 \omega_3^5 + b_1 \omega_4^5 = c_1 \omega^1 + c_2 \omega^2 , \\ &a_2 \omega_3^5 + b_2 \omega_4^5 = c_2 \omega^1 + c_3 \omega^2 , \\ &a_3 \omega_3^5 + b_3 \omega_4^5 = c_3 \omega^1 + c_4 \omega^2 . \end{aligned} \tag{5}$$

It is easy to see that

$$\xi = (a_1 + a_3) v_3 + (b_1 + b_3) v_4 \tag{6}$$

is the mean curvature vector. Our surface being minimal, we have $a_1 + a_3 = b_1 + b_3 = 0$ and (4) + (5) reduce to

$$\begin{aligned} &\omega_1^3 = a_1 \omega^1 + a_2 \omega^2 , \quad \omega_1^4 = b_1 \omega^1 + b_2 \omega^2 , \\ &\omega_2^3 = a_2 \omega^1 - a_1 \omega^2 , \quad \omega_2^4 = b_2 \omega^1 - b_1 \omega^2 , \end{aligned} \tag{7}$$

$$a_1 \omega_3^5 + b_1 \omega_4^5 = c_1 \omega^1 + c_2 \omega^2 , \quad a_2 \omega_3^5 + b_2 \omega_4^5 = c_2 \omega^1 - c_1 \omega^2 . \tag{8}$$

Because of

$$dv_1 = \omega_1^2 v_2 + \omega^1(a_1 v_3 + b_1 v_4) + \omega^2(a_2 v_3 + b_2 v_4) ,$$
$$dv_2 = -\omega_1^2 v_1 + \omega^1(a_2 v_3 + b_2 v_4) - \omega^1(a_1 v_3 + b_1 v_4) ,$$

we have $T^2(M) = \{v_1, v_2, a_1v_3 + b_1v_4, a_2v_3 + b_2v_4\}$ and

(9) $$a_1b_2 - a_2b_1 \neq 0 .$$

From (7),

(10) $$Da_1 \wedge \omega^1 + Da_2 \wedge \omega^2 = 0 , \quad Da_2 \wedge \omega^1 - Da_1 \wedge \omega^2 = 0 ,$$
$$Db_1 \wedge \omega^1 + Db_2 \wedge \omega^2 = 0 , \quad Db_2 \wedge \omega^1 - Db_1 \wedge \omega^2 = 0$$

with

(11) $$Da_1 := da_1 - 2a_2\omega_1^2 - b_1\omega_3^4 = \alpha_1\omega^1 + \alpha_2\omega^2 ,$$
$$Da_2 := da_2 + 2a_1\omega_1^2 - b_2\omega_3^4 = \alpha_2\omega^1 - \alpha_1\omega^2 ,$$
$$Db_1 := db_1 - 2b_2\omega_1^2 + a_1\omega_3^4 = \beta_1\omega^1 + \beta_2\omega^2 ,$$
$$Db_2 := db_2 + 2b_1\omega_1^2 + a_2\omega_3^4 = \beta_2\omega^1 - \beta_1\omega^2 .$$

From (8),

(12) $$(dc_1 - 3c_2\omega_1^2)\wedge\omega^1 + (dc_2 + 3c_1\omega_1^2)\wedge\omega^2 = = (f_1c_1 + f_2c_2)\omega^1\wedge\omega^2 ,$$
$$(dc_2 + 3c_1\omega_1^2)\wedge\omega^1 - (dc_1 - 3c_2\omega_1^2) = = (f_2c_1 - f_1c_2)\omega^1\wedge\omega^2 ,$$

$$(a_1b_2 - a_2b_1)f_1 := \alpha_1b_1 - \alpha_2b_2 - \beta_1a_1 + \beta_2a_2 ,$$
$$(a_1b_2 - a_2b_1)f_2 := \alpha_1b_2 + \alpha_2b_1 - \beta_1a_2 - \beta_2a_1 ,$$

and we get the existence of functions $g_1$ , $g_2$ such that

(13) $dc_1 - 3c_2\omega_1^2 = (g_1 - f_2c_1)\omega^1 + (g_2 - f_1c_1)\omega^2$ ,

$dc_2 + 3c_1\omega_1^2 = (g_2 + f_2c_2)\omega^1 - (g_1 - f_1c_2)\omega^2$ .

In D , consider the isothermic coordinates (u,v) such that

(14) $ds^2 = r^2(du^2 + dv^2)$, $r(u,v) > 0$ ; $\omega^1 = r\,du$, $\omega^2 = r\,dv$ .

Then

(15) $$\omega_1^2 = r^{-1}\,(-\,r_v du + r_u dv)$$

because of $d\omega^1 = -\,\omega^2 \wedge \omega_1^2$ , $d\omega^2 = \omega^1 \wedge \omega_1^2$ . We get

(16) $$\frac{\partial c_1}{\partial u} + 3\,r^{-1}\,r_v c_2 = g_1 r - f_2\,r c_1\,,$$

$$\frac{\partial c_1}{\partial v} - 3\,r^{-1}\,r_u c_2 = g_2 r - f_1\,r c_1\,,$$

$$\frac{\partial c_2}{\partial u} - 3\,r^{-1}\,r_v c_1 = g_2 r + f_2\,r c_2\,,$$

$$\frac{\partial c_2}{\partial v} + 3\,r^{-1}\,r_u c_1 = -\,g_1 r + f_1\,r c_2\,,$$

i.e.,

(17) $$\frac{\partial c_1}{\partial u} + \frac{\partial c_2}{\partial v} + r^{-1}(3r_u + f_2r^2)c_1 + r^{-1}(3r_v - f_1r^2)c_2=0,$$

$$\frac{\partial c_1}{\partial v} - \frac{\partial c_2}{\partial u} + r^{-1}(3r_v + f_1r^2)c_1 - r^{-1}(3r_u - f_2r^2)c_2=0.$$

The function $w := c_1 + ic_2$ is thus a generalized analytic function [1] , and the Theorem follows from the obvious fact $S = \{(u,v);\ w(u,v) = 0\}$ .

R e f e r e n c e

[1] I.N. VEKUA: Verallgemeinerte analytische Funktionen, Berlin,Akademie-Verlag 1963.(Original edition Moskva 1959.)

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