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SEMIGROUP REPRESENTATIONS OF COMMUTATIVE IDEMPOTENT ABELIAN GROUPOIDS

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Praha

Abstract: A general method of obtaining commutative idempotent abelian groupoids is found.

Key words: Groupoid, semigroup, commutative, idempotent, abelian, representation.

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A groupoid is called

- commutative if it satisfies the identity $ab = ba$,
- idempotent if it satisfies the identity $aa = a$,
- abelian if it satisfies the identity $ab.cd = ac.bd$,
- distributive if it satisfies the identities $a.bc = ab.ac$ and $bc.a = ba.ca$.

For the sake of brevity, the commutative idempotent abelian groupoids will be called CIA-groupoids. Clearly, every CIA-groupoid is distributive.

The purpose of this paper is to describe a general method of obtaining CIA-groupoids. We shall prove that a groupoid G is a CIA-groupoid if and only if there exists a uniquely 2-divisible commutative semigroup $S(+)$ such that

$G \subseteq S$ and $xy = 1/2(x + y)$ for all $x,y \in G$. Moreover, we shall show that in general we cannot demand the equality $G = S$.

1. <u>Some properties of distributive groupoids</u>. Let G be a groupoid. A non-empty subset $I \subseteq G$ is an ideal if $ab \in I$ and $ba \in I$, whenever $a \in I$ and $b \in G$. In such a case, we can define a congruence relation r on G as follows:

$$\langle x,y \rangle \in r \text{ iff either } x = y \text{ or } x,y \in I .$$

The corresponding factor-groupoid is denoted by G/I.

If G is a groupoid then $\mathrm{Id}\, G$ will denote the set of all idempotents of G.

1.1. <u>Proposition</u>. Let G be a distributive groupoid. Then

(i) $\mathrm{Id}\, G$ is an ideal of G,

(ii) $a.bc \in \mathrm{Id}\, G$ and $ab.c \in \mathrm{Id}\, G$ for all $a,b,c \in G$.

(iii) The mapping $x \longmapsto x.xx = xx.x$ is a homomorphism of G onto $\mathrm{Id}\, G$,

(iv) $G/\mathrm{Id}\, G$ has just one idempotent.

<u>Proof</u>. (i) We have $x.xx = xx.xx = xx.x$ and $(x.xx)(x.xx) = xx.xx = x.xx$ for all $x \in G$. Thus $x.xx \in \mathrm{Id}\, G$ and $\mathrm{Id}\, G$ is non-empty. If $a \in \mathrm{Id}\, G$ and $b \in G$, then $ab.ab = aa.b = ab$ and $ba.ba = ba$.

(ii) We can write $a.bc = ab.ac = (ab.a)(ab.c) =$

$= (aa.ba)(ab.c) = ((aa.b)(aa.a))(ab.c)$. Since $aa.a \in \mathrm{Id}\, G$ and $\mathrm{Id}\, G$ is an ideal, $a.bc \in \mathrm{Id}\, G$. Similarly, $ab.c \in$ $\in \mathrm{Id}\, G$.

(iii) and (iv) are easy.

1.2. <u>Proposition</u>. The following conditions are equivalent for a groupoid G :

(i) G is distributive and $\mathrm{Id}\, G$ contains just one element.

(ii) There is an element $0 \in G$ such that $a.0 = 0 = 0.a$ and $a.bc = 0 = ab.c$ for all $a,b,c \in G$.

<u>Proof</u>. The proposition is obvious from 1.1.

Every groupoid satisfying the equivalent conditions of 1.2 will be called a BD-groupoid.

1.3. <u>Proposition</u>. Let G be a distributive groupoid. Then $\mathrm{Id}\, G$ is an idempotent distributive groupoid and $G/\mathrm{Id}\, G$ is a BD-groupoid. Moreover, G is isomorphic to a subdirect product of $\mathrm{Id}\, G$ and $G/\mathrm{Id}\, G$.

<u>Proof</u>. Apply 1.1 and 1.2.

1.4. <u>Proposition</u>. Let $f: G \to H$ be a homomorphism of distributive groupoids. Then f induces two homomorphisms $g: \mathrm{Id}\, G \to \mathrm{Id}\, H$ and $h: G/\mathrm{Id}\, G \to H/\mathrm{Id}\, H$. Moreover, if f is injective (surjective) then both g and h are so .

<u>Proof</u>. An easy work.

A groupoid is called triabelian if every its subgroupoid generated by three (not necessarily different) elements is abelian.

1.5. <u>Proposition</u>. (i) A distributive groupoid is triabelian iff it satisfies the identity $ab.ca = ac.ba$.

(ii) Every commutative distributive groupoid is triabelian.

(iii) Every distributive cancellation groupoid is triabelian.

Proof. (i) The "only if" part of the proof is obvious. For the "if" part we can assume that G is idempotent (due to 1.3 and to the fact that every BD-groupoid is abelian).

First suppose that a,b,c,d are four elements of G satisfying $ab.cd = ac.bd$. Denote by $S(a,b,c,d)$ the subgroupoid generated by a,b,c,d . As it is easy to see, the set $\{x \mid ab.xd = ax.bd\}$ is a subgroupoid containing a,b,c,d , and hence $ab.xd = ax.bd$ for all $x \in S(a,b,c,d)$. Quite similarly we can prove that $ay.xd = ax.yd$ for all $x,y \in S(a,b,c,d)$.

Now let $a,b,c \in G$ be arbitrary elements. We are going to show that the subgroupoid $S(a,b,c)$ generated by a,b,c is abelian. Since $ab.cb = ac.bb$ and $ab.cc = ac.bc$, $ax.yb = ay.xb$ and $ax.yc = ay.xc$ for all $x,y \in S(a,b,c)$. The subgroupoid $\{z \mid ax.yz = ay.xz\}$ contains thus the elements a,b,c and we get $ax.yz = ay.xz$ for all $x,y,z \in S(a,b,c)$. From the reason of symmetry, $bx.yz = by.xz$ and $cx.yz = cy.xz$, and consequently $ux.yz = uy.xz$ for all $u,x,y,z \in S(a,b,c)$.

(ii) This assertion is an immediate consequence of (i).

(iii) It is proved in [1] that every distributive quasigroup generated (as a quasigroup) by three elements is abelian. According to [5], every distributive cancellation groupoid is a subgroupoid of a distributive quasigroup and the assertion follows easily.

Remark. The following problem seems to be open. Is every distributive groupoid triabelian?

A congruence r of a groupoid G is called normal if the factor G/r is a cancellation groupoid. A groupoid is called ideal-simple if it has no proper ideal.

1.6. Proposition. The following conditions are equivalent for a commutative distributive groupoid G :

(i) G is ideal-simple.

(ii) Every congruence of G is normal.

Proof. (i) implies (ii). By 1.1, G is idempotent. Let r be a congruence of G, $\langle ab,ac \rangle \in r$ for some $a,b,c \in G$ and $I = \{x \mid \langle xb,xc \rangle \in r\}$. As one may check easily, I is an ideal and therefore $I = G$. Thus $\langle bb,bc \rangle \in r$ and $\langle cb,cc \rangle \in r$, and so $\langle b,c \rangle \in r$.

1.7. Corollary. (i) Every ideal-simple commutative distributive groupoid is an idempotent cancellation groupoid.

(ii) Every commutative distributive division groupoid is a quasigroup.

(iii) Every congruence of a commutative distributive quasigroup is normal.

Example. Let Q be the set of all rational numbers and $x * y = 2x - y$ for all $x,y \in Q$. It is easy to check that Q is an abelian idempotent quasigroup with respect to $*$ and the relation r, defined by $\langle x,y \rangle \in r$ iff $x - y$ is an integer, is a congruence of $Q(*)$. However, r is not normal, since $\langle 1/2 * 1, 0 * 1 \rangle \in r$. The factorgroupoid

$Q(*)/r$ is an ideal-simple distributive division groupoid which is not cancellative.

Let H be a subgroupoid of a groupoid G. We say that a congruence s of G is an extension of a congruence r of H if $r = s \cap (H \times H)$.

1.8. Proposition. Let G be a subgroupoid of a distributive quasigroup Q such that G is contained in no proper subquasigroup of Q. Then every normal congruence of G can be extended to exactly one normal congruence of Q.

Proof. Apply Zorn's lemma for a more detailed proof, see [4].

2. Congruences of free CIA-groupoids. Denote by R the set of all rational numbers $2^{-m}.c$, where c is an integer and m is a natural number. For every $n \geq 1$, the cartesian power R^n is a CIA-quasigroup with respect to the operation $\circ$ defined by

$$\langle a_1,\ldots,a_n \rangle \circ \langle b_1,\ldots,b_n \rangle = \langle 1/2(a_1 + b_1),\ldots \\ \ldots,1/2(a_n + b_n) \rangle .$$

In fact (see [3]), R^n is a free CIA-quasigroup and the elements $e_0^n = \langle 0,0,\ldots,0 \rangle$, $e_1^n = \langle 1,0,\ldots,0 \rangle$, $e_2^n = \langle 0,1,0,\ldots,0 \rangle$, ..., $e_n^n = \langle 0,\ldots,0,1 \rangle$ are its generators. Let F_n denote the set of all $\langle a_1,\ldots,a_n \rangle \in R^n$ satisfying $a_1 \geq 0,\ldots,a_n \geq 0$ and $a_1 + \ldots + a_n \leq 1$. Clearly, F_n is a subgroupoid of R^n. As it is proved in [3], F_n is a free CIA-groupoid and the elements $e_0^n, e_1^n,\ldots,e_n^n$

are its free generators. Further define

$F_{n,i} = \{\langle a_1,\dots,a_n\rangle \in F_n \mid a_i = 0\}$ for $i = 1,2,\dots,n$,

$F_{n,*} = \{\langle a_1,\dots,a_n\rangle \in F_n \mid a_1 +\dots+ a_n = 1\}$,

$H_i = F_n \setminus F_{n,i}$ for $i = 1,2,\dots,n$,

$H_0 = F_n \setminus F_{n,*}$ and

$\mathrm{Int}\, F_n = F_n \setminus (F_{n,1} \cup F_{n,2} \cup \dots \cup F_{n,n} \cup F_{n,*})$.

It is not difficult to show that all these sets are subgroupoids of F_n . Moreover, $\mathrm{Int}\, F_n$ is ideal-simple and it is an ideal of F_n .

The following lemma is proved in [4]. However, the proof is easy.

2.1. <u>Lemma</u>. Let A be a subgroupoid of F_n with $\mathrm{Int}\, F_n \subseteq A$ and r be a congruence of A . If $\langle a, a\circ b\rangle \in r$ for some $a \in A$ and $b \in \mathrm{Int}\, F_n$, then $\langle a,b\rangle \in r$.

2.2. <u>Lemma</u>. Let r be a congruence of F_n and $a, b \in H_0$. Then $\langle a,b\rangle \in r$ iff $\langle 1/2a, 1/2b\rangle \in r$.

<u>Proof</u>. The direct implication is easy. To prove the converse, first assume that $\langle a_1,\dots,a_n\rangle = a \in \mathrm{Int}\, F_n$. Since $\langle b_1,\dots,b_n\rangle = b \in H_0$, $b_1 +\dots+ b_n < 1$. For every $i = 1,2,\dots$, let

$$c_i = \langle 2^{-i}a_1 + (1 - 2^{1-i})b_1,\dots,2^{-i}a_n + (1 - 2^{1-i})b_n\rangle ,$$

$$d_i = \langle (1 - 2^{-i})b_1,\dots,(1 - 2^{-i})b_n\rangle ,$$

$$p_i = \langle (1 + 2^{-i})b_1,\dots,(1 + 2^{-i})b_n\rangle \text{ and}$$

$$q_i = \langle 2^{1-i}a_1 + (1 - 2^{1-i})b_1,\dots,2^{1-i}a_n + (1 - 2^{1-i})b_n\rangle .$$

As it is easy to see, the following equalities hold:

$c_1 = \langle 1/2a_1,\dots,1/2a_n\rangle = 1/2a$, $d_1 = 1/2b$, $q_1 = a$,

$c_{i+1} = c_i \circ b$, $d_{i+1} = d_i \circ b$, $q_{i+1} = q_i \circ b$, $c_i \in F_n$,

$p_i \in R^n$, $q_i \in \mathrm{Int}\, F_n$, $d_i \circ p_i = b$ and $c_i \circ p_i = q_{i+2}$.

Since $b_1 + \dots + b_n < 1$, there is $k \geq 1$ such that $p_k \in F_n$. Then $\langle c_1,d_1\rangle \in r$, $\langle c_2,d_2\rangle \in r,\dots, \langle c_k,d_k\rangle \in r$, $\langle c_k \circ p_k, d_k \circ p_k\rangle \in r$, and hence $\langle q_{k+2},b\rangle \in r$. Several applications of 2.1 (for $A = F_n$) give $\langle q_{k+1},b\rangle \in r$, $\langle q_k,b\rangle \in r,\dots, \langle q_2,b\rangle \in r$ and $\langle a,b\rangle \in r$.

In the general case we shall proceed by induction on n . If $n = 1$ then either $a = b$ or $a \in \mathrm{Int}\, F_1$ or $b \in \mathrm{Int}\, F_1$ (since $F_{1,*} = \{1\}$ and $F_{1,1} = \{0\}$). Let $n \geq 2$. If there exists an $i \in \{1,\dots,n\}$ such that both a and b belong to $F_{n,i}$, then the induction hypothesis can be applied, since $F_{n,i}$ is (canonically) isomorphic to F_{n-1} . If no such an i exists then $a \circ b \in \mathrm{Int}\, F_n$. As $\langle 1/2a, 1/2b\rangle \in r$, we have $\langle 1/2a, 1/2a \circ 1/2b\rangle \in r$, and consequently $\langle a, a \circ b\rangle \in r$. Quite similarly $\langle a \circ b, b\rangle \in r$, and so $\langle a,b\rangle \in r$.

2.3. Lemma. Let r be a congruence of F_n and $i \in \{0,1,\dots,n\}$. Define a relation r_i on F_n by $\langle a,b\rangle \in r_i$ iff $\langle e_i^n \circ a, e_i^n \circ b\rangle \in r$. Then r_i is a congruence of F_n .

Proof. The lemma is obvious.

2.4. Lemma. Let r be a congruence of F_n . Then

(i) $r \subseteq r_i$ for all $i \in \{0,1,\dots,n\}$,

(ii) $r_i \cap (H_i \times H_i) = r \cap (H_i \times H_i)$ for all $i \in \{0,1,\dots$ $\dots,n\}$,

(iii) $r = r_0 \cap r_1 \cap \dots \cap r_n$.

<u>Proof</u>. (i) is obvious. By 2.2, (ii) holds for $i = 0$. Let $i \in \{1,\dots,n\}$. Define an automorphism f of F_n by $f(\langle x_1,\dots,x_n\rangle) = \langle x_1,\dots,x_{i-1}, 1 - (x_1 + \dots + x_n),$ $x_{i+1},\dots,x_n\rangle$. Clearly, $f = f^{-1}$, $f(e_0^n) = e_i^n$, $f(e_i^n) =$ $= e_0^n$ and $f(e_j^n) = e_j^n$ for all $j \notin \{0,i\}$. Put $\langle a,b\rangle \in s$ iff $\langle f(a),f(b)\rangle \in r$. Obviously, s is a congruence of F_n and $s_0 \cap (H_0 \times H_0) = s \cap (H_0 \times H_0)$. If $\langle a,b\rangle \in r_i \cap$ $\cap (H_i \times H_i)$ then $\langle e_i^n \circ a, e_i^n \circ b\rangle \in r$, $\langle f(e_i^n \circ a), f(e_i^n \circ b)\rangle \in$ $\in s$, $\langle e_0^n \circ f(a), e_0^n \circ f(b)\rangle \in s$, $\langle f(a),f(b)\rangle \in s_0$, $\langle f(a),f(b)\rangle \in s_0 \cap (H_0 \times H_0) \subseteq s$ and $\langle a,b\rangle \in r \cap$ $\cap (H_i \times H_i)$. Thus $r_i \cap (H_i \times H_i) \subseteq r \cap (H_i \times H_i)$.

Finally we shall prove (iii). By (i), $r \subseteq r_0 \cap r_1 \cap$ $\cap \dots r_n$. Let $\langle a,b\rangle \in r_0 \cap r_1 \cap \dots \cap r_n$. As one may check easily, there exist two numbers $j,k \in \{0,1,\dots,n\}$ such that both a and $a \circ b$ belong to H_j and both b and $a \circ b$ belong to H_k. We have $\langle a, a \circ b\rangle \in (r_0 \cap r_1 \cap$ $\cap \dots \cap r_n) \cap (H_j \times H_j) \subseteq r_j \cap (H_j \times H_j) = r \cap (H_j \times H_j) \subseteq$ $\subseteq r$ and similarly $\langle a \circ b, b\rangle \in r_k \cap (H_k \times H_k) \subseteq r$. Thus $\langle a,b\rangle \in r$.

3. <u>Semigroup representations of CIA-groupoids</u>. A commutative semigroup $G(+)$ is called uniquely 2-divisible

if the mapping $x \mapsto x + x$ is a permutation of G. The inverse permutation is denoted by $1/2x$.

3.1. Proposition. Let $G(+)$ be a uniquely 2-divisible commutative semigroup. Define a binary operation on G by $xy = 1/2(x + y)$. Then the groupoid G is a CIA-groupoid.

Proof. The proposition is obvious.

If G is a CIA-groupoid and there exists a binary operation $+$ on G such that $G(+)$ is a uniquely 2-divisible commutative semigroup and $xy = 1/2(x + y)$ for all x, $y \in G$, then the semigroup $G(+)$ will be called a semigroup representation of G. We shall denote by $\mathcal{A}$ the class of all CIA-groupoids which have a semigroup representation.

3.2. Proposition. Let $G(+)$ be a semigroup representation of a CIA-groupoid G. Then

(i) G is a cancellation groupoid iff $G(+)$ is a cancellation semigroup,

(ii) G is a quasigroup iff $G(+)$ is a group,

(iii) the semigroup $G(+)$ has a unit element (i.e., an element e such that $x + e = x$ for all x) iff there is an $a \in G$ such that $x \mapsto xa$ is a permutation.

Proof. We shall prove only the converse implication of (iii), since the rest is obvious. There exists $e \in G$ with $ea = 1/2a$. For any $x \in G$, it is $(x + e).a = 1/2(x + e + + a) = 1/2(x + 2(ea)) = 1/2(x + a) = xa$, so that $x + e = = x$.

Let us denote by $\mathcal{B}$ the class of all CIA-groupoids which have a semigroup representation with a unit element.

3.3. <u>Proposition</u>. A CIA-groupoid G belongs to $\mathcal{B}$ iff there is an $a \in G$ such that $x \mapsto xa$ is a permutation.

<u>Proof</u>. It suffices to put $x + y = f(xy)$, where $f(za) = z$.

<u>Remark</u>. $\mathcal{B} \subseteq \mathcal{A}$ and $\mathcal{B} \neq \mathcal{A}$. The subgroupoid $G = \{x \mid x \in R, x > 0\}$ of the quasigroup R belongs to $\mathcal{A} \setminus \mathcal{B}$. It follows from 3.2 and 3.3 that every semigroup representation of $H \in \mathcal{B}$ has a unit element.

3.4. <u>Proposition</u>. (i) Every finite subdirectly irreducible CIA-groupoid is contained in $\mathcal{B}$.

(ii) Every finite CIA-groupoid is a subgroupoid of a finite CIA-groupoid from $\mathcal{B}$.

(iii) If $G \in \mathcal{A}$ ($G \in \mathcal{B}$) and r is a normal congruence of G, then $G/r \in \mathcal{A}$ ($G/r \in \mathcal{B}$).

(iv) If $G \in \mathcal{A}$ ($G \in \mathcal{B}$) and r is a fully invariant congruence of G, then $G/r \in \mathcal{A}$ ($G/r \in \mathcal{B}$).

<u>Proof</u>. (i) Let G be a finite subdirectly irreducible CIA-groupoid. If $a,x,y \in G$ then $\langle x,y \rangle \in s_a$ means $xa = ya$. Obviously, s_a is a congruence relation for every a and $\bigcap_{a \in G} s_a = id_G$. Consequently there is an $a \in G$ such that $xa = ya$ iff $x = y$. Since G is finite, the mapping $x \mapsto xa$ is a permutation and we may use 3.3.

(ii) It follows from (i).

(iii) Let r be a normal congruence of a CIA-groupoid G with a semigroup representation G(+). If $\langle x,y \rangle \in r$ then $z.(6z.2x) = 2z + 1/2x = 4z.x$, $z.(6z.2y) = 4z.y$ and $\langle 4z.x, 4z.y \rangle \in r$ for all $z \in G$. Since r is normal,

$\langle 2x,2y \rangle \in r$. Conversely, if $\langle 2x,2y \rangle \in r$, then $\langle 4x,4y \rangle \in r$, $\langle 2x.4x, 2y.4y \rangle \in r$, $3x = 2x.4x$, $3y = 2y.4y$, $3x.x = 2x$, $3y.y = 2y$, and so $\langle x,y \rangle \in r$. The remainder of the proof is clear.

(iv) Similarly.

3.5. Proposition. Non-trivial free CIA-groupoids have no semigroup representations.

Proof. With respect to 3.4 (iii) and to the fact that F_1 is a cancellation groupoid, it suffices to show that $F_1 \notin \mathcal{A}$. Suppose, on the contrary, that F_1 has a semigroup representation $F_1(*)$. Let $f(x) * f(x) = x$ for each $x \in F_1$, $a = 1/2 * 1/2$ and $b = 1 - a$. Clearly, $a, b \in F_1$, $1/2 = a \circ b = f(a) * f(b)$ and $a = 1/2 * 1/2 = f^{-1}(1/2) = a * b$. By 3.2 (i), $F_1(*)$ is a cancellation semigroup. Since $x * a = x * b * a$ for every $x \in F_1$, b is a unit element of $F_1(*)$. Thus $F_1 \in \mathcal{B}$, a contradiction with 3.3.

3.6. Theorem. Every CIA-groupoid is a subgroupoid of a CIA-groupoid with a semigroup representation.

Proof. The class $\mathcal{C}$ of all subgroupoids of groupoids from $\mathcal{A}$ is closed under ultraproducts. Since every groupoid is isomorphic to an ultraproduct of its finitely generated subgroupoids (see e.g. [2]), it is enough to prove that every finitely generated CIA-groupoid belongs to $\mathcal{C}$. In other words, we must prove that $F_n/r \in \mathcal{C}$ for every $n \geq 1$ and every congruence r of F_n . As F_n/r is isomorphic to a subdirect product of F_n/r_0 , F_n/r_1 ,..., F_n/r_n (by 2.4) and $\mathcal{C}$ is closed under cartesian products and subgroupoids, it remains to show that $F_n/r_i \in C$ for all $i \in \{0,1,\dots,n\}$. We may restrict ourselves to the case $i = 0$, since the rôles of free generators are symmetrical. The set $A = \{\langle x_1,\dots,x_n\rangle \mid x_j \in R,\ x_j \geq 0\}$ is a uniquely 2-divisible commutative semigroup with respect to the usual addition. Define a binary relation s on A as follows:

$$\langle x,y\rangle \in s \quad \text{iff} \quad \langle 2^{-k}x, 2^{-k}y\rangle \in r \cap (H_0 \times H_0) \text{ for}$$

some k.

As it is easy to see, s is a congruence of the semigroup A and $\langle x,y\rangle \in s$ iff $\langle 1/2x, 1/2y\rangle \in s$. By 2.4, $r_0 = s \cap (F_n \times F_n)$ and the factorsemigroup A/s is a semigroup representation of F_n/r_0.

3.7. <u>Corollary</u>. A groupoid G is a CIA-groupoid iff there exists a uniquely 2-divisible commutative semigroup $S(+)$ such that $G \subseteq S$ and $xy = 1/2(x + y)$ for all $x, y \in G$.

3.8. <u>Corollary</u>. For every commutative abelian distributive groupoid G there exist a commutative semigroup $S(+)$ and an automorphism f of $S(+)$ such that $G \subseteq S$ and $xy = f(x + y)$ for all $x,y \in G$.

<u>Proof</u>. Apply 1.2, 1.3 and 3.7.

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