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Jan Pelant

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REMARK ON LOCALLY FINE SPACES

Jan PELANT, Praha

Abstract: Locally fine coreflection is constructed in [I] by an iterative method. The first locally fine approximation $M(\mathcal{U})$ of a uniform space $(X,\mathcal{U})$ is defined as follows: $M(\mathcal{U}) = \{\{O_\iota \cap P^\iota_\alpha\}_{\iota,\alpha} \mid \{O_\iota\} \in \mathcal{U}$ and $\{P^\iota_\alpha\} \in \mathcal{U}$ for each $\iota\}$. The first locally fine approximation will be called a derivative in the present remark. It is shown that a derivative of uniformity need not be a uniformity.

Key words: Uniform spaces, locally fine coreflection, point-finite base.

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Introduction: It is one of unsolved problems of [I] whether a derivative of each uniform space forms a uniformity. Some answers are given in [I], e.g. a derivative of a unifom space with a point-finite (or σ-disjoint) base forms a uniformity. A derivative was used in [I] for the construction of locally fine coreflection. Because of difficulties with the proof that the derivative is a uniformity, the notion of a quasiuniformity was introduced. Hence we are going to show that it was necessary to do it. Our main proposition is : X is a uniform space. A derivative of X^m is a uniformity for each cardinal m iff X has a point-finite base. This assertion is not useless as we believe that examples of uniform spaces without point-finite base are given in [P].

It is my pleasant duty to thank Z. Frolík who turned my attention to this problem and P. Simon whose simplifications are used in the proofs of the present note.

<u>Definition</u>: Let $(X,\mathcal{U})$ be a uniform space. Morita's derivative $M(\mathcal{U})$ of $(X,\mathcal{U})$ is defined as follows:
$M(\mathcal{U}) = \{\{O_\iota \cap P^\iota_\alpha\}_{\iota,\alpha} \mid \{O_\iota\} \in \mathcal{U}, \forall \iota : \{P^\iota_\alpha\} \in \mathcal{U}\}$.

<u>Proposition</u>: $(X,\mathcal{U})$ is a uniform space. The following conditions are equivalent:

1) $(X,\mathcal{U})$ has a base of point-finite covers.

2) $\forall \mathcal{P} \in \mathcal{U}\ \exists\ f: X \longrightarrow \mathcal{P} : (f(x) \ni x$ for each $x)$

$$\exists\ \mathcal{Q} \in \mathcal{U}\ \forall \mathcal{R} < \mathcal{Q},\ \mathcal{R} \in \mathcal{U}\ \forall R \in \mathcal{R} :$$

$:$ card $f(R) < \omega_0$.

<u>Remark</u>. 2) is stated in [P] in fact.

<u>Proof</u>: $1 \Longrightarrow 2$. $\mathcal{P} \in \mathcal{U}$. Take any $\mathcal{S} \in \mathcal{U}$ which is uniformly locally finite and refines $\mathcal{P}$ (it is possible, see [I]). Suppose that $\mathcal{S}$ is well-ordered. Define $f': X \longrightarrow \mathcal{S}$ by $f'(x) = \min\{S \in \mathcal{S} \mid x \in S\}$. Choose a mapping φ : $: \mathcal{S} \longrightarrow \mathcal{P}$ such that $\varphi(S) \supset S$ for each $S \in \mathcal{S}$. Define $f = \varphi \circ f'$. Any uniform cover, each member of which meets only a finite number of members of $\mathcal{S}$, can play the role of $\mathcal{Q}$ from 2).

$2 \Longrightarrow 1$. It is sufficient to prove this implication for metric spaces only. Choose $\varepsilon > 0$. By the assumption, there is a partition $\mathcal{D}$ of X such that all classes of $\mathcal{D}$ have a diameter less than $\frac{\varepsilon}{3}$ and there is δ, $\frac{\varepsilon}{6} > \delta > 0$ such that $B_\delta(x)$ intersects only finitely many classes of

$\mathcal{D}$ for each $x \in X$. $B_\delta(x)$ is δ-ball with a center in $x \in X$; $B_\delta(Z) = \bigcup \{B_\delta(x) \mid x \in Z\}$ for $Z \subset X$. It means that the cover $\{B_\delta(D)\}_{D \in \mathcal{D}}$ is point-finite. Clearly, $\operatorname{diam} B_\delta(D) < \varepsilon$ for each $D \in \mathcal{D}$. Hence $\{B_\delta(D)\}_{D \in \mathcal{D}} < \{B_\varepsilon(x)\}_{x \in X}$. QED.

Theorem. Let $(X, \mathcal{U})$ be a uniform space that has not any base of point-finite covers. Let m be a cardinal greater than cardinality of any uniform cover of $\mathcal{U}$. Then a derivative of $(X, \mathcal{U})^m$ is not a uniformity.

Proof: By Proposition, $(X, \mathcal{U})$ satisfies: $\exists \mathcal{P} \in \mathcal{U}\ \forall f: X \longrightarrow \mathcal{P} : f(x) \ni x\ \forall \mathcal{Q} \in \mathcal{U}\ \exists \mathcal{R} < \mathcal{Q}$,

$$\mathcal{R} \in \mathcal{U}\ \exists\ R \in \mathcal{R} : \operatorname{card} f(R) \geq \omega_0 .$$

Take such a wild $\mathcal{P}$. Choose $i_0 \in m$. Take some one-to-one mapping $K: \mathcal{P} \longrightarrow m - \{i_0\}$. We are going to define a cover $\mathcal{Z}$ of a derivative of X^m: $\mathcal{Z} = \{\pi_{i_0}^{-1}(P) \cap \pi_{K(P)}^{-1}(Q) \mid P, Q \in \mathcal{P}\}$. To spare space denote $[Z] = P$ for $Z \in \mathcal{Z}$ with $Z = \pi_{i_0}^{-1}(P) \cap \pi_{K(P)}^{-1}(Q)$.

Suppose there is $\mathcal{W} \in M(\mathcal{U}^m)$ such that $\mathcal{W} \overset{*}{<} \mathcal{Z}$. We may suppose that $\mathcal{W}$ is of the form:

$\mathcal{W} = \{\pi_{i_0}^{-1}(R) \cap \bigcap_{\substack{i \in I_R \\ T_i \in \mathcal{T}^R}} \pi_i^{-1}(T_i) \mid R \in \mathcal{R} \in \mathcal{U},\ \mathcal{T}^R \in \mathcal{U}$

for each $R \in \mathcal{R}$, I_R is a finite subset of $m\}$.

Choose a mapping $F: X^m \longrightarrow \mathcal{Z}$ such that $st(y, \mathcal{W}) \subset F(y)$ for each $y \in X$. Let us observe that $I_R \supset \{K([F(y)]) \mid y \in \pi_{i_0}^{-1}(R)\}$ for each $R \in \mathcal{R}$.

Define $f: X \longrightarrow \mathcal{P}$ by $f(x) = [F(\xi_x)]$, $\pi_i(\xi_x) = x$ for each $i \in m$.

There is $R_0 \in \mathcal{R}$ such that card $f(R_0) \geq \omega_0$. As K is one-to-one, it holds: card $\{K([F(y)]) \mid y \in \pi_{i_0}^{-1}(R_0)\} \geq$ $\geq$ card $\{K(f(x)) \mid x \in R_0\} \geq \omega_0$.

Hence we have found even two infinite subsets of the finite set I_{R_0} which is a contradiction.

<u>Corollary 1</u>. $(X, \mathcal{U})$ has a point-finite base iff a derivative of $(X, \mathcal{U})^m$ is a uniformity for each cardinal m.

<u>Proof</u>: For "if only" part see [I].

<u>Corollary 2</u>. If $\mathcal{K}$ is a productive class of uniform spaces such that a derivative of each member of $\mathcal{K}$ is a uniformity, then each member of $\mathcal{K}$ has a point-finite base.

<u>Corollary 3</u>. Let $(X, \mathcal{U})$ be a uniform space. If X has a σ-disjoint base, then X has a point-finite base as well.

<u>Proof</u>: A derivative of any uniform space with σ-disjoint base forms a uniformity (see [I], p. 142).

References

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Matematický ústav
Karlova universita
Sokolovská 83, 18600 Praha 8
Československo

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