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Václav Slavík

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ON H-PRIMITIVE LATTICES

Václav SLAVÍK, Praha

Abstract: This paper is concerned with h-primitive lattices. There are shown infinitely many primitive classes of lattices which are h-characterizable by means of a single lattice and are not characterizable.

Key words: Primitive class, splitting lattice, projective lattice, characterizable class, h-characterizable class.

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Given a set E of finite lattices, we shall denote by $N(E)$ the class of all lattices that contain no sublattice isomorphic to a lattice in E and by $N_h(E)$ ($N_{hf}(E)$) the class of all lattices L such that no homomorphic image of any sublattice (finite sublattice) of L belongs to E. A class K of lattices will be called characterizable (h-characterizable, hf-characterizable) if there exists a set E of finite lattices such that $K = N(E)$ ($K = N_h(E)$, $K = N_{hf}(E)$). If E is a finite set of finite lattices $\{L_1, \dots, L_n\}$, the classes $N(E)$, $N_h(E)$ and NHf(E) will be denoted by $N(L_1, \dots, L_n)$, $N_h(L_1, \dots, L_n)$ and $N_{hf}(L_1, \dots, L_n)$, respectively. A finite lattice L is said to be primitive (see [3]) (h-primitive, hf-primitive) if the class $N(L)$ ($N_h(L)$, $N_{hf}(L)$) is primitive. It is evident that $N(E) \supseteq N_{hf}(E) \supseteq N_h(E)$. If $K = N(E)$ is a characterizable primitive class of lattices and a lattice L does not belong to $N_h(E)$ then there exists a homomorphism of a sublattice S of L onto a lattice A in E. Since $A \notin K$, $S \notin K$ and $L \notin K$. We have proved that K is hf-characterizable and h-characterizable and $K = N(E) = N_{hf}(E) = N_h(E)$. Similarly we can prove that any hf-characterizable primitive class of lattices $K = N_{hf}(E)$ is h-characterizable and $K = N_{hf}(E) = N_h(E)$. Especially, any primitive lattice L is hf-primitive and $N(L) = N_{hf}(L)$; any hf-primitive lattice L is h-primitive and $N_{hf}(L) = N_h(L)$. If K is an hf-characterizable primitive class of lattices, then a lattice L belongs to K iff any finite sublattice of L belongs to K and thus K is characterizable (see [1]). The purpose of the present paper is to show that there exist h-primitive lattices that are not hf-primitive, hf-primitive lattices that are not primitive and h-characterizable primitive classes of lattices that are not characterizable. Notice that Igošin ([2]) has shown that there exist h-characterizable primitive classes of algebras with one unary operation that are not characterizable.

McKenzie ([5]) investigates splitting lattices, i.e. finite subdirectly irreducible lattices B such that there exists an equation $p = q$ and any primitive class K of lattices satisfies precisely one of the following conditions: either K satisfies $p = q$ or $B \in K$.

<u>Theorem 1</u>. A finite lattice B is h-primitive if and only if B is a splitting lattice.

<u>Proof</u>. Let B be an h-primitive lattice. The class

$N_h(B)$ is finitely based ([2]) and thus it can be characterized by an equation $p = q$. Let K be a primitive class of lattices that does not satisfy the equation $p = q$. Then there is a lattice $L \in K$, $L \notin N_h(B)$. Since B is a homomorphic image of a sublattice of L, $B \in K$. The equation $p = q$ is not satisfied in B and thus B is a splitting lattice. But if B is a splitting lattice, then there exists (see [5]) a homomorphism f of $FL(k)$, the free lattice with k generators, onto B and $p, q \in FL(k)$ such that $\mathrm{Ker}\, f$ is the greatest congruence of $FL(k)$ that separates p, q. We shall show that $N_h(B)$ is the class of all lattices satisfying the equation $p = q$. If $L \notin N_h(B)$, then there is a homomorphism of a sublattice of L onto B and since $p = q$ is not satisfied in B, the equation $p = q$ is not satisfied in L. If a lattice L does not satisfy $p = q$, then there exists a homomorphism h of $FL(k)$ into L such that $h(p) \neq h(q)$. Since $\mathrm{Ker}\, h \subseteq \mathrm{Ker}\, f$, there exists a homomorphism g of L into B such that $g \circ h = f$ and thus $L \notin N_h(B)$. So the class $N_h(B)$ is primitive, i.e. B is h-primitive.

Theorem 2. Let B be a finite lattice. The following conditions are equivalent:

(1) B is hf-primitive.

(2) B is h-primitive and $N_h(B)$ is characterizable.

(3) B is subdirectly irreducible and there exists a homomorphism f of a finite sublattice L of a free lattice onto B.

Moreover, if B is hf-primitive, then $N_{hf}(B) = N_h(B) = N(E)$, where E is the set of all lattices A such that there exist homomorphisms g of L onto A and h of A onto B with $h \circ g = f$.

<u>Proof</u>. Assume (1). Then B is evidently h-primitive, $N_h(B) = N_{hf}(B)$ and since any hf-characterizable primitive class is characterizable, we have (2). Suppose (2). Let $N_h(B) = N(E)$. Since B is a homomorphic image of some $FL(k)$, $FL(k)$ does not belong to $N_h(B) = N(E)$, there exists a finite sublattice C of $FL(k)$ isomorphic to a lattice in E. $C \notin N(E) = N_h(B)$ and we get that there is a homomorphism of a sublattice L of C onto B. It is evident that L is a sublattice of $FL(k)$. Clearly, any h-primitive lattice must be subdirectly irreducible. Now, assume (3). McKenzie ([5]) has shown that any finite subdirectly irreducible lattice which is a homomorphic image of a free lattice is a splitting lattice, i.e, h-primitive. We shall show that $N_h(E) = N_{hf}(B)$. Clearly, $N_h B \subseteq N_{hf}(B)$. If a lattice $S \notin N_h(B)$, there exists a homomorphism h of a sublattice C of S onto B. Since L is projective ([4],[5]), there is a homomorphism g of L into C such that $h \circ g = f$. Since $g(L)$ is a finite sublattice of S, $S \notin N_{hf}(B)$. Thus $N_h(B) = N_{hf}(B)$ and so B is hf-primitive. The proof can now be finished easily.

Given a finite lattice L, define a lattice L^* in this way: L is a sublattice of L^*, $L^* \setminus L$ contains exactly three elements u, v, a; u is the smallest and v the greatest element of L^* and a is comparable with no element of L.

<u>Theorem 3</u>. Let L be a h-primitive lattice. Then L^*

is h-primitive, too. Moreover, the following holds:

(1) If $N_h(L)$ is the class of all lattices satisfying an equation $p(x_1,\dots,x_n) = q(x_1,\dots,x_n)$, then $N_h(L^*)$ is the class of all lattices satisfying the equation $p^*(x_1,\dots,x_{n+1}) = q^*(x_1,\dots,x_{n+1})$, where $p^*(x_1,\dots,x_{n+1}) = p(t_1,\dots,t_n)$, $q^*(x_1,\dots,x_{n+1}) = q(t_1,\dots,t_n)$ and $t_k = (x_k \wedge i) \vee o$ ($k = 1,2,\dots,n$), $o = (x_1 \wedge \dots \wedge x_n) \vee (x_{n+1} \wedge (x_1 \vee \dots \vee x_n))$, $i = (x_1 \vee \dots \vee x_n) \wedge (x_{n+1} \vee (x_1 \wedge \dots \wedge x_n))$.

(2) L^* is hf-primitive iff L is hf-primitive.

(3) L^* is primitive iff L is primitive.

Proof. Let $N_h(L)$ be the class of all lattices satisfying the equation $p(x_1,\dots,x_n) = q(x_1,\dots,x_n)$. Let $a_1,\dots,a_n$, d be elements of a lattice S such that $p^*(a_1,\dots,a_n,d) \neq q^*(a_1,\dots,a_n,d)$. Put $r = (a_1 \vee \dots \vee a_n) \wedge (d \vee (a_1 \wedge \dots \wedge a_n))$, $s = (a_1 \wedge \dots \wedge a_n) \vee (d \wedge (a_1 \vee \dots \vee a_n))$, $\ell_k = (a_k \wedge r) \vee s$ ($k = 1,2,\dots,n$). Since $p^*(a_1,\dots,a_n,d) = p(\ell_1,\dots,\ell_n)$ and $q^*(a_1,\dots,a_n,d) = q(\ell_1,\dots,\ell_n)$, the equation $p = q$ is not satisfied in the interval $[s,r]$. There exists a homomorphism f of a sublattice S' of $[s,r]$ onto L. Since $d \wedge r = d \wedge s$ and $d \vee r = d \vee s$, the set $S' \cup \{d, d \wedge r, d \vee r\}$ forms a sublattice of S that can be homomorphically mapped onto L^*. Thus $S \notin N_h(L^*)$. The equation $p = q$ is not satisfied in L and thus there exist elements $a_1,\dots,a_n$ of L such that $p(a_1,\dots,a_n) \neq q(a_1,\dots,a_n)$. Clearly, $p^*(a_1,\dots,a_n,a) = p(a_1,\dots,a_n)$ and $q^*(a_1,\dots,a_n,a) = q(a_1,\dots,a_n)$. The equation $p^* = q^*$ is not satisfied in L and we get that any lattice satisfying $p^* = q^*$ belongs to $N_h(L^*)$. It is easy to show that L is a homomorphic image of a finite sublattice of a free lattice (L is a sublattice of a free lattice) iff L^* has the same property.

Now we shall show that there exist h-primitive lattices that are not hf-primitive.

<u>Lemma</u> 1. For any positive integer n , the lattice B_n in Fig. 1 is generated by the elements a,b,c , and there exists a homomorphism f_n of B_n onto the lattice B_o in Fig. 1 such that $f_n(a) = a$, $f_n(b) = b$, $f_n(c) = c$.

<u>Proof</u>. It is easy to verify that the elements o,d,e,f, k,h,g,ℓ,p,i,r,s,t,u,v , are in the sublattice C of B_n generated by $\{a,b,c\}$. Since $t_1 = b \vee \ell$, $v_1 = \ell \vee c$, $s_1 = a \vee \ell$, $z_1 = s_1 \wedge v_1$, $u_1 = t_1 \wedge u$, we have $\{s_1,t_1,u_1,v_1,z_1\} \subseteq C$. Assume $\{s_i,t_i,u_i,v_i,z_i\} \subseteq C$. Since $s_i \vee u_i = s_{i+1}$, $v_i \vee u_i = v_{i+1}$, $s_{i+1} \wedge v_{i+1} = z_{i+1}$, $b \vee z_{i+1} = t_{i+1}$ and $t_{i+1} \wedge u = u_{i+1}$, we have $\{s_{i+1},t_{i+1},u_{i+1},v_{i+1},z_{i+1}\} \subseteq C$. Thus we get that $C = B_n$. One can easily verify that the mapping f_n of B_n into B_o defined by $f_n(s_k) = s$, $f_n(t_k) = t$, $f_n(u_k) = f_n(z_k) = u$, $f_n(v_k) = v$ for all k , $1 \leq k \leq n$, and $f_n(x) = x$ for all other $x \in B_n$, is a homomorphism of B_n onto B_o such that $f_n(a) = a$, $f_n(b) = b$, $f_n(c) = c$.

<u>Theorem 4</u>. The lattice B_o in Fig.1 is h-primitive and it is not hf-primitive.

<u>Proof</u>. McKenzie ([5]) has shown that B_o is a splitting lattice, i.e., by Theorem 1, B_o is h-primitive. Suppose that B_o is hf-primitive. By Theorem 2, there exists a homomorphism f of a sublattice C of a free lattice onto B_o . Since C

is projective ([4],[5]), there exist homomorphisms g_n of C into B_n such that $f_n \circ g_n = f$. There exist elements a', b', c' of C such that $g_n(a') = a$, $g_n(b') = b$, $g_n(c') = c$. Thus g_n are homomorphism of C onto B_n and so C cannot be finite; a contradiction.

Corollary 1. Any finite sublattice of a free lattice satisfies the inclusion

$$(a \vee (b \wedge c))(b \vee (a \wedge c))(c \vee (a \wedge b)) \leq (a \wedge (b \vee c)) \vee (b \wedge (a \vee c)) \vee \vee (c \wedge (a \vee b)) .$$

Proof. All finite sublattices of a free lattice belong to $N_h(B_o)$ and $N_h(B_o)$ is the class of all lattices satisfying this inclusion (see [5]).

Starting from the lattice B_o in Fig. 1, we can obtain by Theorem 3 an infinite sequence of h-primitive lattices that are not hf-primitive. Hereby we obtain infinitely many h-characterizable primitive classes of lattices that are not characterizable.

Finally we shall give a construction of hf-primitive lattices that are not primitive.

Let A be the lattice given in Fig. 1 and let L be a primitive lattice (i.e. a finite subdirectly irreducible sublattice of a free lattice) of cardinality greater than two. Define a lattice $A(L)$ in this way: $A(L) = A \cup L$, A and L are sublattices of $A(L)$, $x \wedge y = x \wedge a = x \wedge c$ and $x \vee y = x \vee a = x \vee c$ for all $x \in A$, $y \in L$.

Lemma 2. The lattice $A(L)$ is a sublattice of a free lattice.

Proof. We shall show that $A(L)$ is projective. Let f

be a homomorphism of a lattice S onto $A(L)$. Since A is projective (see [3],[5]), there exists a sublattice A' of S such that $f|_{A'}$ is an isomorphism of A' onto A. Let $a' \in A'$ and $b' \in A'$ be such that $f(a') = a$ and $f(b') = b$. If $c \in S$ and $f(c) \in L$, then $f((c \vee b') \wedge a) = f(c)$. The interval $[b', a']$ is mapped by f onto L. The lattice L is projective and thus there exists a sublattice L of $[b', a']$ such that $f|_{L'}$ is an isomorphism of L' onto L. The set $A' \cup L'$ forms a sublattice of S and $f|_{A' \cup L'}$ is an isomorphism of $A' \cup L'$ onto $A(L)$.

If we identify in $A(L)$ the greatest element v of L with a and the smallest element u of L with b, we get a subdirectly irreducible lattice $B(L)$ that is a homomorphic image of $A(L)$. Since v is join reducible, i.e. there are $v_1, v_2 \in L$ such that $v \neq v_1$, $v \neq v_2$, $v = b_1 \vee v_2$, we get $v_1 \vee v_2 = e \wedge f$ in $B(L)$ and since $e \wedge f \nleq v_1$, $e \wedge f \nleq v_2$, $e \nleq v_1 \vee v_2$, $f \nleq v_1 \vee v_2$ in $B(L)$, the lattice $B(L)$ is not a sublattice of a free lattice. Using Theorem 2 we obtain

<u>Theorem 5</u>. The lattice $B(L)$ is hf-primitive and $B(L)$ is not primitive.

Since the lattices L_n ($n = 1,2,\ldots$) in Fig. 1 are primitive (see [3],[5]) we have that lattices $B(L_n)$ ($n = 1,2,\ldots$) are hf-primitive and $B(L_n)$ are not primitive. Using Theorem 3 we can obtain other examples of such lattices.

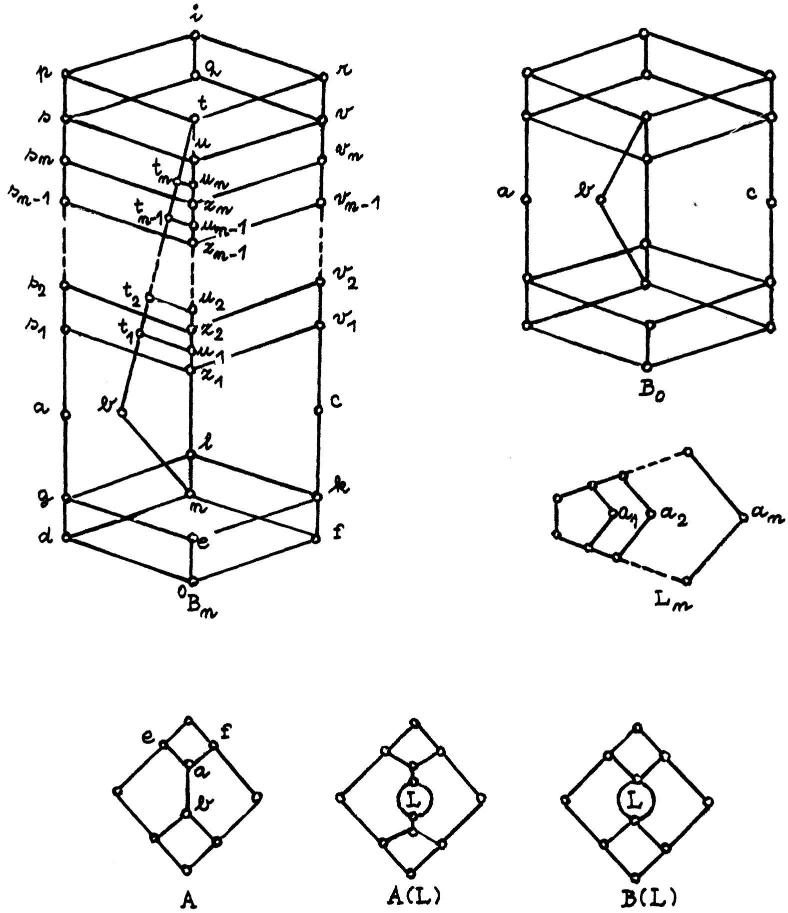

Fig. 1

R e f e r e n c e s

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Matematicko-fyzikální fakulta

Karlova universita

Sokolovská 83, 18600 Praha 8

Československo

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