

Josef Jirásko

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GENERALIZED INJECTIVITY

J. JIRÁSKO, Praha

Abstract: In this paper, a new general theory of injectivity of left R -modules is introduced. The existence and unicity of the injective envelope of every module is established for a large class of injectivities. Some earlier known results on injectivities with respect to preradicals are derived from the theory in a more general form.

Key-words: \mathcal{L} -injective module, \mathcal{L} -injective envelope, preradical.

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We start with some basic definitions and notations. Throughout this paper, R stands for an associative ring with unit element and $R\text{-mod}$ denotes the category of all unitary left R -modules. If $f \in \text{Hom}_R(N, M)$ and P is a submodule of M then $f^{-1}(P) = \{x \in N, f(x) \in P\}$. The fact that A is an essential submodule of B (i.e. A meets every nonzero submodule of B in a nonzero submodule) will be denoted by $A \subseteq' B$.

A class \mathcal{Q} of modules is said to be abstract if it is closed under isomorphisms, hereditary if it is abstract and closed under submodules and cohereditary if it is closed under homomorphic images.

§ 1. General theory. In the class $\mathcal{M} = \{ \langle M, N, f, Q \rangle, M, N, Q \in R\text{-mod}, N \subseteq M, f \in \text{Hom}_R(N, Q) \}$ define the partial order \leq in the following way:
 $\langle M, N, f, Q \rangle \leq \langle M', N', f', Q' \rangle$ if and only if $M = M', N \subseteq N', Q = Q'$ and $f|_N = f'$.

In this paragraph \mathcal{L} always denotes a subclass of \mathcal{M} . The following five conditions on \mathcal{L} will be useful later.

- (α) $\langle M, N, f, Q \rangle \in \mathcal{L}, \langle M, N', f', Q \rangle \in \mathcal{M}, \langle M, N, f, Q \rangle \leq \langle M, N', f', Q \rangle$ implies $\langle M, N', f', Q \rangle \in \mathcal{L}$,
 (β) $\langle M, N, f, A \rangle \in \mathcal{L}, A \xrightarrow{i} B$ implies $\langle M, N, if, B \rangle \in \mathcal{L}$,
 (β') $\langle M, N, f, A \rangle \in \mathcal{L}, A \xrightarrow{i} B, A \subseteq B$ implies $\langle M, N, if, B \rangle \in \mathcal{L}$,
 (γ) $\langle M, N, f, A \rangle \in \mathcal{L}, A \xrightarrow{g} B$ an isomorphism, implies $\langle M, N, gf, B \rangle \in \mathcal{L}$,
 (σ) $\langle M, N, f, A \rangle \in \mathcal{L}, A \xrightarrow{g} B$ implies $\langle M, N, gf, B \rangle \in \mathcal{L}$.

For every $\langle B, A, f, Q \rangle \in \mathcal{M}$ let us define $r_{\mathcal{L}}(B, A, f, Q)$ ($s_{\mathcal{L}}(B, A, f, Q)$) to be a submodule of B generated by all the $g(M), g \in \text{Hom}_R(M, B)$, to which there exists a commutative diagram

$$\begin{array}{ccc}
 N & \xrightarrow{\quad} & M \\
 h \downarrow & & \downarrow g \\
 A & \xrightarrow{\quad} & B \\
 f \downarrow & & \\
 & & Q
 \end{array}$$

with $\langle M, N, fh, Q \rangle \in \mathcal{L}$ ($N = g^{-1}(A)$).

We also use the following abbreviations: $r_{\mathcal{L}}(B, Q, 1_Q, Q) = r_{\mathcal{L}}(B, Q)$, $s_{\mathcal{L}}(B, Q, 1_Q, Q) = s_{\mathcal{L}}(B, Q)$, $r_{\mathcal{L}}(\hat{Q}, Q) = r_{\mathcal{L}}(Q)$, $s_{\mathcal{L}}(\hat{Q}, Q) = s_{\mathcal{L}}(Q)$.

Lemma 1.1. Let \mathcal{L} be a subclass of \mathcal{M} and

$$\begin{array}{ccc}
 & & f \\
 & Q \longleftarrow & A \longrightarrow B \\
 (*) \quad k \downarrow & & \downarrow l' \quad \downarrow l \\
 & T \xrightarrow{h} & C \longrightarrow D \\
 & & (**) \quad g \downarrow \quad \downarrow g' \\
 & & P \longrightarrow M \\
 & & A \longrightarrow B
 \end{array}$$

commutative diagrams. If for every diagram (**), $\langle M, P, kfg, T \rangle \in \mathcal{L}$ whenever $\langle M, P, fg, Q \rangle \in \mathcal{L}$ then $l(r_{\mathcal{L}}(B, A, f, Q)) \subseteq r_{\mathcal{L}}(D, C, h, T)$. Especially, $l(r_{\mathcal{L}}(B, Q)) \subseteq r_{\mathcal{L}}(D, T)$.

Proof: Obvious.

Definition 1.2. We say that a module Q is \mathcal{L} -injective, if every diagram

$$(1) \quad \begin{array}{ccc}
 & N \longrightarrow & M \\
 & \downarrow f & \\
 & Q &
 \end{array}$$

with $\langle M, N, f, Q \rangle \in \mathcal{L}$ can be completed to a commutative one.

Theorem 1.3. Consider the following five conditions concerning a module Q :

- (i) Q is a direct summand in each extension $N \supseteq Q$ such that $N \subseteq Q + r_{\mathcal{L}}(\hat{N}, Q)$,
- (ii) $Q \supseteq r_{\mathcal{L}}(Q)$,
- (iii) every diagram (1) with $M \subseteq N + r_{\mathcal{L}}(\hat{M}, N, f, Q)$ can be made commutative,
- (iv) Q is \mathcal{L} -injective,
- (v) $Q \supseteq s_{\mathcal{L}}(Q)$.

Then the conditions (i), (ii), (iii) are equivalent and (iii) implies (iv) and (iv) implies (v). Moreover, if \mathcal{L} satisfies (∞) then all the five conditions are equivalent.

Proof: (i) implies (ii). The module $N = Q + r_{\mathcal{L}}(Q)$ is an essential extension of Q so that $N = Q$ by (i).

(ii) implies (iii). Consider the diagram (1) and extend f to $g: \hat{M} \rightarrow \hat{Q}$. Then $g(M) \subseteq g(N) + g(r_{\mathcal{L}}(\hat{M}, N, f, Q)) \subseteq Q + r_{\mathcal{L}}(Q) = Q$.

(iii) implies (i). Obvious.

(iv) implies (v). Take the commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{i} & M \\ h \downarrow & & \downarrow g \\ Q & \xrightarrow{j} & \hat{Q} \end{array}$$

where $N = g^{-1}(Q)$ and $\langle M, N, h, Q \rangle \in \mathcal{L}$.

Then $h = fi$ for some $f: M \rightarrow Q$ and $g = jf$ since $(g - jf)(M) \cap Q = 0$, as it is easily seen.

(ii) implies (iv). Obvious.

Finally, if \mathcal{L} satisfies (∞) then $r_{\mathcal{L}} = s_{\mathcal{L}}$ and we are through.

Corollary 1.4: Let \mathcal{L} be a subclass of \mathcal{M} satisfying (β) . If $P, Q \in R\text{-mod}$, $P \subseteq Q$, then P is \mathcal{L} -injective provided $P \supseteq r_{\mathcal{L}}(Q)$.

Proof: We have $r_{\mathcal{L}}(P) \subseteq r_{\mathcal{L}}(Q)$ by Lemma 1.1 and P is \mathcal{L} -injective by Theorem 1.3.

Definition 1.5. For any ordinal α and modules $A \subseteq B$ let us define the sequence $r_{\mathcal{L}}^{\alpha}(\hat{B}, A)$ of modules inductively as follows:

$$r_{\mathcal{L}}^0(\hat{B}, A) = A$$

$$r_{\mathcal{L}}^{\alpha+1}(\hat{B}, A) = r_{\mathcal{L}}^{\alpha}(\hat{B}, A) + r_{\mathcal{L}}(\hat{B}, r_{\mathcal{L}}^{\alpha}(\hat{B}, A))$$

$$\text{and } r_{\mathcal{L}}^{\alpha}(\hat{B}, A) = \bigcup_{\beta < \alpha} r_{\mathcal{L}}^{\beta}(\hat{B}, A), \quad \alpha \text{ limit.}$$

Further, put $\bar{r}_{\mathcal{L}}(\hat{B}, A) = r_{\mathcal{L}}^{\alpha}(\hat{B}, A)$ where $r_{\mathcal{L}}^{\alpha}(\hat{B}, A) = r_{\mathcal{L}}^{\alpha+1}(\hat{B}, A)$.

Corollary 1.6: For every module Q , the module $\bar{r}_{\mathcal{L}}(\hat{Q}, Q)$ is \mathcal{L} -injective.

Proof: Denote $P = \bar{r}_{\mathcal{L}}(\hat{Q}, Q)$. Then $r_{\mathcal{L}}(P) = r_{\mathcal{L}}(\hat{Q}, P) \subseteq \bar{r}_{\mathcal{L}}(\hat{Q}, Q) = P$ and apply Theorem 1.3.

Lemma 1.7. Let \mathcal{L} be a subclass of \mathcal{M} satisfying (γ) and $A \subseteq B$ be modules. If $f: \hat{B} \rightarrow \tilde{B}$ is a B -isomorphism of two injective envelopes \hat{B}, \tilde{B} of B then $f(r_{\mathcal{L}}^{\alpha}(\hat{B}, A)) = r_{\mathcal{L}}^{\alpha}(\tilde{B}, A)$.

Proof: It follows easily by transfinite induction using Lemma 1.1.

Lemma 1.8. Let \mathcal{L} be a subclass of \mathcal{M} satisfying (β') . If $Q \subseteq P \subseteq \hat{Q}$ then $r_{\mathcal{L}}^{\alpha}(\hat{Q}, Q) \subseteq r_{\mathcal{L}}^{\alpha}(\hat{Q}, P)$ for every ordinal α .

Proof: By transfinite induction and Lemma 1.1.

Theorem 1.9. Let \mathcal{L} be a subclass of \mathcal{M} satisfying $(\alpha), (\beta')$. Then $\bar{r}_{\mathcal{L}}(\hat{Q}, Q)$ is the smallest \mathcal{L} -injective submodule of \hat{Q} containing Q .

Proof: The module $\bar{r}_{\mathcal{L}}(\hat{Q}, Q)$ is \mathcal{L} -injective by Corollary 1.6. Let a module $P, Q \subseteq P \subseteq \hat{Q}$, be \mathcal{L} -injective. From Theorem 1.3 we get $P = r_{\mathcal{L}}^1(\hat{Q}, P)$ and Lemma 1.8 then yields $\bar{r}_{\mathcal{L}}(\hat{Q}, Q) \subseteq P$.

Definition 1.10. An \mathcal{L} -injective module B is said to be an \mathcal{L} -injective envelope of a module A if there is no proper \mathcal{L} -injective submodule of B containing A .

Remark 1.11: If \mathcal{L} satisfies (γ) then the class of \mathcal{L} -injective modules is abstract.

Theorem 1.12. If the subclass \mathcal{L} of \mathcal{M} satisfies

$(\alpha), (\beta)$ and (γ) then every module Q has an \mathcal{L} -injective envelope which is unique up to Q -isomorphism.

Proof: The existence of an \mathcal{L} -injective envelope of Q follows from Theorem 1.9. Put $Q_\alpha = r_{\mathcal{L}}^\alpha(\hat{Q}, Q)$ and let f_0 be the canonical embedding $Q \hookrightarrow P$ where P is an arbitrary \mathcal{L} -injective envelope of Q . Suppose that $f_\alpha : Q_\alpha \rightarrow P$ is a monomorphism. From $Q_{\alpha+1} = Q_\alpha + r_{\mathcal{L}}(\hat{Q}, Q_\alpha)$ we get $Q_{\alpha+1} \subseteq Q_\alpha + r_{\mathcal{L}}(\hat{Q}, Q_\alpha, f_\alpha, P)$ by Lemma 1.1 (using $(\beta), (\gamma)$) and consequently f_α extends to a homomorphism $f_{\alpha+1} : Q_{\alpha+1} \rightarrow P$ by Theorem 1.3. It is easy to see that $f_{\alpha+1}$ is a monomorphism. From this, one easily derives the existence of a monomorphism $f : \bar{Q} = \bar{r}_{\mathcal{L}}(\hat{Q}, Q) \rightarrow P$ extending the identity on Q . Hence $f(\bar{Q})$ is \mathcal{L} -injective by 1.11, so that $f(\bar{Q}) = P$, which finishes the proof.

Definition 1.13. A submodule A of a module B is said to be \mathcal{L} -dense in B if $B \subseteq A + r_{\mathcal{L}}(\hat{B}, A)$. An essential \mathcal{L} -dense submodule A of B is said to be \mathcal{L} -essential.

Theorem 1.14. If \mathcal{L} satisfies (α) then a module Q is \mathcal{L} -injective if and only if it has no proper \mathcal{L} -essential extensions.

Proof: The condition is necessary since every \mathcal{L} -dense extension of an \mathcal{L} -injective module splits by Theorem 1.3. Conversely, $r_{\mathcal{L}}^1(\hat{Q}, Q)$ is an \mathcal{L} -essential extension of Q so that $r_{\mathcal{L}}(Q) \subseteq Q$ and Q is \mathcal{L} -injective by Theorem 1.3.

Definition 1.15. Let \mathcal{L} be a subclass of \mathcal{M} satisfying (γ) . A module A is said to be weakly \mathcal{L} -dense in B

if $B \subseteq \overline{F}_{\mathcal{L}}(\widehat{B}, A)$. An essential weakly \mathcal{L} -dense submodule A of B is said to be weakly \mathcal{L} -essential.

Theorem 1.16. If \mathcal{L} satisfies (α) and (γ) then a module Q is \mathcal{L} -injective if and only if it has no proper weakly \mathcal{L} -essential extensions.

Proof: The sufficiency follows from Theorem 1.14. Conversely, suppose that K is a weakly \mathcal{L} -essential extension of Q . Then $r_{\mathcal{L}}^1(\widehat{Q}, Q) = Q$ by Theorem 1.3 and so $Q \subseteq \subseteq K \subseteq \overline{F}_{\mathcal{L}}(\widehat{K}, Q) = \overline{F}_{\mathcal{L}}(\widehat{Q}, Q) = Q$.

Remark 1.17: If \mathcal{L} satisfies (γ) then $\overline{F}_{\mathcal{L}}(\widehat{Q}, Q)$ is the greatest weakly \mathcal{L} -dense extension of Q contained in \widehat{Q} since for a weakly \mathcal{L} -dense extension P of Q with $Q \subseteq P \subseteq \widehat{Q}$ we have $P \subseteq \overline{F}_{\mathcal{L}}(\widehat{P}, Q) = \overline{F}_{\mathcal{L}}(\widehat{Q}, Q)$.

Lemma 1.18. Let \mathcal{L} satisfy (β') and (γ) and $A \subseteq B \subseteq \subseteq C$ be modules. Then

- (i) if A is a weakly \mathcal{L} -essential submodule of C then B is weakly \mathcal{L} -essential in C ,
- (ii) if A is weakly \mathcal{L} -essential in B and B weakly \mathcal{L} -essential in C then A is weakly \mathcal{L} -essential in C .

Proof: (i) is immediate since Lemma 1.1 yields $\overline{F}_{\mathcal{L}}(\widehat{C}, A) \subseteq \overline{F}_{\mathcal{L}}(\widehat{C}, B)$. Further, we have $B \subseteq \overline{F}_{\mathcal{L}}(\widehat{B}, A)$, $C \subseteq \subseteq \overline{F}_{\mathcal{L}}(\widehat{C}, B)$ and Lemma 1.1 gives $C \subseteq \overline{F}_{\mathcal{L}}(\widehat{C}, B) \subseteq \overline{F}_{\mathcal{L}}(\widehat{C}, \overline{F}_{\mathcal{L}}(\widehat{C}, A)) = \overline{F}_{\mathcal{L}}(\widehat{C}, A)$.

Theorem 1.19. The following are equivalent for a class \mathcal{L} satisfying (α) , (β) and (γ) :

- (i) N is a maximal weakly \mathcal{L} -essential extension of Q ,
- (ii) N is an \mathcal{L} -injective envelope of Q ,
- (iii) N is \mathcal{L} -injective weakly \mathcal{L} -essential extension of Q .

Proof: (i) implies (ii). The \mathcal{L} -injectivity of N follows from 1.16. If K is \mathcal{L} -injective, $Q \subseteq K \subseteq N$ then 1.18 and 1.16 yield $K = N$.

(ii) implies (iii). It follows from Theorem 1.9 and 1.12 that $Q \subseteq N$. Hence $Q \subseteq N \subseteq \hat{Q} = \hat{N}$ and $N = \overline{F}_{\mathcal{L}}(\hat{Q}, Q)$ by Theorem 1.9.

(iii) implies (i). By Theorem 1.16.

Proposition 1.20. Let \mathcal{L} be a subclass of \mathcal{M} satisfying $(\alpha), (\sigma)$ and Q be a module. Then Q is \mathcal{L} -injective if and only if every diagram (1) with N \mathcal{L} -dense in M can be made commutative.

Proof: If the condition is satisfied then $Q = r_{\mathcal{L}}^1(\hat{Q}, Q)$ and Q is \mathcal{L} -injective by Theorem 1.3. Conversely, $M \subseteq N + r_{\mathcal{L}}(\hat{M}, N)$ and it suffices to use Theorem 1.3 (iii) since $r_{\mathcal{L}}(\hat{M}, N) \subseteq r_{\mathcal{L}}(\hat{M}, N, f, Q)$ by Lemma 1.1.

§ 2. $(\mathcal{A}, \mathcal{B})$ -injective modules.

Definition 2.1. Let \mathcal{A} and \mathcal{B} be non-empty classes of modules. We say that a module Q is $(\mathcal{A}, \mathcal{B})$ -injective if every diagram (1) with $M/N \in \mathcal{A}$ and $M/\text{Ker } f \in \mathcal{B}$ can be made commutative.

Baer's lemma 2.2. If \mathcal{A} and \mathcal{B} are abstract, hereditary and cohereditary classes of modules then a module Q is $(\mathcal{A}, \mathcal{B})$ -injective if and only if for every left ideal I of R with $R/I \in \mathcal{A}$ every homomorphism $f: I \rightarrow Q$ with $R/\text{Ker } f \in \mathcal{B}$ can be extended to $g: R \rightarrow Q$.

Proof: We proceed to the sufficiency, the necessity being obvious. Suppose that there is a diagram (1) with

$M/N \in \mathcal{A}$ and $M/\text{Ker } f \in \mathcal{B}$ which cannot be made commutative. By Zorn's lemma, we can assume that f cannot be extended to any $N \not\subseteq K \subseteq M$. Let $b \in M \setminus N$ be arbitrary, $I = (N:b)$. Then $R/I \cong (Rb + N)/N$ lies in \mathcal{A} . Further, defining $\varphi : I \rightarrow Q$ by $\varphi(r) = f(rb)$ we have $\text{Ker } \varphi = (\text{Ker } f : b)$ and consequently $R/\text{Ker } \varphi \cong (Rb + \text{Ker } f)/\text{Ker } f$ lies in \mathcal{B} . Thus φ extends to $\psi : R \rightarrow Q$ and hence f extends to $g : \langle N, b \rangle \rightarrow Q$ given by $g(n + rb) = f(n) + \psi(r)$, a contradiction.

§ 3. Applications. Let \mathcal{P} be a subclass of the class \mathcal{R} of all couples (M, N) , $N \subseteq M$. We say that \mathcal{P} satisfies the condition (a) if $(M, N) \in \mathcal{P}$, $N \subseteq N' \subseteq M$ implies $(M, N') \in \mathcal{P}$.

Remark 3.1: Let \mathcal{K} , \mathcal{P} be subclasses of \mathcal{R} and $\mathcal{L} = \{ \langle M, N, f, Q \rangle ; (M, N) \in \mathcal{P}, f \in \text{Hom}_R(N, Q), (M, \text{Ker } f) \in \mathcal{K} \}$. Obviously, \mathcal{L} satisfies (β) and (γ) . Moreover, if both \mathcal{K} and \mathcal{P} satisfy (a) then \mathcal{L} satisfies (α) and (σ) .

Now we recall some basic definitions from the theory of preradicals (for details see [4] and [5]).

A preradical s for $R\text{-mod}$ is any subfunctor of the identity functor, i.e. s assigns to each module M its submodule $s(M)$ in such a way that every homomorphism of M into N induces a homomorphism of $s(M)$ into $s(N)$ by restriction. A preradical is said to be

- idempotent if $s(s(M)) = s(M)$ for every module M ,
- a radical if $s\left(\frac{M}{s(M)}\right) = 0$ for every module M ,

- hereditary if $s(N) = N \cap s(M)$ for every submodule N of a module M .

A module M is s -torsion if $s(M) = M$ and s -torsionfree if $s(M) = 0$. If r and s are preradicals then we write $r \leq s$ if $r(M) \subseteq s(M)$ for all $M \in R\text{-mod}$. The zero functor is denoted by zer and the identity functor by id .

For every $M \in R\text{-mod}$ we define $r_{\{M\}}(N) = \sum f(M)$, f ranging over all $f \in \text{Hom}_R(M, N)$. It is easy to see that $r_{\{M\}}$ is an idempotent preradical and, in fact, the smallest preradical for which M is a torsion module.

For a preradical s and modules $N \subseteq M$ let us define $C_s(N:M)$ by $C_s(N:M)/N = s(\hat{M}/N)$. If $M = \hat{N}$ then we write simply $C_s(N) = C_s(N:\hat{N})$. Obviously, for $N_0 \subseteq N$, $M_0 \subseteq M$ and $f \in \text{Hom}_R(M, N)$ with $f(M_0) \subseteq N_0$ we have $f(C_s(M_0:M)) \subseteq C_s(N_0:N)$.

Definition 3.2. Let s and u be preradicals for $R\text{-mod}$. A submodule N of a module M is said to be s_u -dense in M if $M \subseteq C_s(N:C_u(M))$. A preradical s is said to be balanced if $A/B \cong C/D$ implies that B is s_{id} -dense in A if and only if D is s_{id} -dense in C .

Remark 3.3: The fact that N is s_{id} -dense in M means that N is s -dense in M in the sense of Beachy [1]. Further, N is s_{zer} -dense in M if and only if \hat{M}/N is s -torsion and if s is hereditary then s_u -density means the same as s_{zer} -density for every preradical u .

Lemma 3.4. Let s and u be preradicals for $R\text{-mod}$

and $N_1 \subseteq N_2 \subseteq N$ be modules. If N_1 is s_u -dense in N then N_2 is so.

Proof: Obvious since $N \subseteq C_s(N_1 : C_u(N)) \subseteq C_s(N_2 : C_u(N))$.

Definition 3.5. Let to every $M \in R\text{-mod}$ correspond four preradicals $s^{(M)}, t^{(M)}, u^{(M)}, v^{(M)}$. Let \mathcal{P} be the class of all couples (M, N) of modules such that N is $s_{u^{(M)}}^{(M)}$ -dense in M and \mathcal{X} be the class of all couples (M, N) such that N is $t_{v^{(M)}}^{(M)}$ -dense in M . Now let \mathcal{L} be the class of all $\langle M, N, f, Q \rangle$ such that $(M, N) \in \mathcal{P}$, $f \in \text{Hom}_R(N, Q)$ and $(M, \text{Ker } f) \in \mathcal{X}$. We say that a module Q is (s, t, u, v) -injective if it is \mathcal{L} -injective.

Proposition 3.6. Every module Q has an (s, t, u, v) -injective envelope which is unique up to Q -isomorphism.

Proof: Both classes \mathcal{P} and \mathcal{X} satisfy Condition (a) by Lemma 3.4 so that it suffices to use Remark 3.1 and Theorem 1.12.

Lemma 3.7. Let s, t, u, v be preradicals for $R\text{-mod}$, A, B, M be modules, $A \subseteq B$ and $f \in \text{Hom}_R(M, \hat{B})$ be such that $f^{-1}(A)$ is s_u -dense in M and $\text{Ker } f$ is t_v -dense in M . Then $f(M) \subseteq C_s(A : \hat{B}) \cap t(\hat{B})$.

Proof: Easy.

Lemma 3.8. Let $A \subseteq B$ be modules and s, t, u, v preradicals for $R\text{-mod}$ satisfying one of the following conditions:

- (i) $u = t = \text{id}$, A is s_{id} -dense in B ,
- (ii) s is idempotent and $u = \text{zer}$, $t = \text{id}$,
- (iii) s is hereditary and $v = \text{id}$.

If, in the notation of 3.5, $s^{(M)} = s$, $t^{(M)} = t$,

$u^{(M)} = u, v^{(M)} = v$ for every $M \in R\text{-mod}$ then $C_{\mathfrak{g}}(A:\hat{B}) \cap t(\hat{B}) \subseteq r_{\mathfrak{g}}(\hat{B}, A)$.

Proof: Put $M = C_{\mathfrak{g}}(A:\hat{B}) \cap t(\hat{B})$ and $N = A \cap t(\hat{B})$. It is easy to see that N is s_u -dense in M and 0 is t_v -dense in M from which the assertion follows easily.

Definition 3.9. Let s, t be preradicals for $R\text{-mod}$. We say that A is an (s, t) -dense submodule of B if $B \subseteq A + (C_{\mathfrak{g}}(A:\hat{B}) \cap t(\hat{B}))$. An essential, (s, t) -dense submodule A of B is said to be (s, t) -essential in B .

Proposition 3.10. Let s, t be preradicals for $R\text{-mod}$ and $A \subseteq B$ be modules. Then A is (s, t) -dense in B if and only if A is s_{id} -dense in B and $B = A + (B \cap t(\hat{B}))$.

Proof: Easy.

We say that the preradicals s, t, u, v for $R\text{-mod}$ satisfy Condition $(*)$ if one of the following holds:

- (i) $u = t = id$,
- $(*)$ (ii) $u = zer, t = id$ and s is idempotent,
- (iii) $v = id$ and s is hereditary.

Corollary 3.11: Under the notation of 3.5 let $s^{(M)} = s, t^{(M)} = t, u^{(M)} = u, v^{(M)} = v$ for every $M \in R\text{-mod}$. If s, t, u, v satisfy Condition $(*)$ and $A \subseteq B$ are modules then A is \mathcal{L} -dense in B if and only if A is (s, t) -dense in B .

Proof: The proof of the necessity is direct and the sufficiency follows immediately from 3.8.

Corollary 3.12: The following are equivalent for preradicals s, t, u, v for $R\text{-mod}$ satisfying Condition $(*)$:

- (i) Q is a direct summand in each extension N in which

it is (s,t) -dense,

(ii) $Q \supseteq C_s(Q) \cap t(\hat{Q})$,

(iii) every diagram (1) with N (s,t) -dense in M can be made commutative,

(iv) Q is (s,t,u,v) -injective.

(See [11, 2.5.]

Proof: Conditions (iii) and (iv) are equivalent by 1.20 and 3.11. Further, by 3.11, Condition (i) means the same as that of Theorem 1.3. Now $r_{\mathcal{L}}(Q) = C_s(Q) \cap t(\hat{Q})$ by 3.7 and 3.8 and Theorem 1.3 finishes the proof since \mathcal{L} satisfies Condition (α) by Lemma 3.4.

Corollary 3.13: Let s, t, u, v be preradicals for R -mod satisfying Condition $(*)$. For any $Q \in R$ -mod define the sequence of modules Q_α inductively as follows: $Q_0 = Q$, $Q_{\alpha+1} = Q_\alpha + (C_s(Q_\alpha : \hat{Q}) \cap t(\hat{Q}))$ and $Q_\alpha = \bigcup_{\beta < \alpha} Q_\beta$, α limit. Then the module $\bar{Q} = Q_\alpha$ where $Q_\alpha = Q_{\alpha+1}$ is the smallest (s,t,u,v) -injective submodule of \hat{Q} containing Q .

Proof: By Lemma 3.7, 3.8 and Theorem 1.9.

Lemma 3.14. Let t be a preradical, s a radical and $A \subseteq B \subseteq C$ be modules. If A is (s,t) -essential in B and B is (s,t) -essential in C then A is (s,t) -essential in C .

Proof: With respect to Proposition 3.10 it suffices to show that if A is s_{id} -dense in B and B is s_{id} -dense in C then A is s_{id} -dense in C . But this follows easily from the radical property of s .

Corollary 3.15: Let s, t, u, v be preradicals for R -mod satisfying Condition $(*)$. If s is a radical and

$Q \in R\text{-mod}$ then $\bar{Q} = Q + (C_s(Q:\hat{Q}) \cap t(\hat{Q}))$ is the smallest (s,t,u,v) -injective submodule of \hat{Q} containing Q .
(See [1], 2.7.)

Proof: In the notation of Corollary 3.13, Q is (s,t) -dense in Q_2 by Lemma 3.14, so that $Q_2 = Q_1$ and Corollary 3.13 completes the proof.

Corollary 3.16: Let s,t,u,v be preradicals for $R\text{-mod}$ satisfying Condition $(*)$, $Q \in R\text{-mod}$. The module Q is (s,t,u,v) -injective if and only if it has no proper (s,t) -essential extension.

Proof: By Corollary 3.11 and Theorem 1.14.

Corollary 3.17: Let s,t,u,v be preradicals for $R\text{-mod}$ satisfying Condition $(*)$, s be a radical and $Q, N \in R\text{-mod}$. The following are equivalent:

- (i) N is a maximal (s,t) -essential extension of Q ,
- (ii) N is an (s,t,u,v) -injective envelope of Q ,
- (iii) N is an (s,t,u,v) -injective (s,t) -essential extension of Q .

Proof: It follows immediately from Lemmas 3.7, 3.8, 3.14 and Theorem 1.19.

Corollary 3.18 (Baer's lemma). Let r, s be hereditary preradicals for $R\text{-mod}$. Then a module Q is (s,t,zer,zer) -injective if and only if for every left ideal I s_{zer} -dense in R every homomorphism $f: I \rightarrow Q$ with $\text{Ker } f$ t_{zer} -dense in R can be extended to $g: R \rightarrow Q$.

Proof: By Lemma 2.2.

Lemma 3.19. Let s be a hereditary preradical for $R\text{-mod}$ and t be a balanced preradical. If $A \subseteq B$ are modules,

$f \in \text{Hom}_R(M, C_s(A:\hat{B}) \cap t(\hat{B}))$, $g = if$ where i is the inclusion of $C_s(A:\hat{B}) \cap t(\hat{B})$ in B , then $g^{-1}(A)$ is s_{zer} -dense in M and $\text{Ker } g$ is t_{id} -dense in M .

Proof: Easy.

Corollary 3.20: Let to every $M \in R\text{-mod}$ correspond a hereditary preradical $s^{(M)}$, a balanced preradical $t^{(M)}$, $u^{(M)} = \text{zer}$, $v^{(M)} = \text{id}$ and let \mathcal{L} be as in 3.5. If $A, B \in R\text{-mod}$, $A \subseteq B$ then $r_{\mathcal{L}}(\hat{B}, A) = \sum_M r_{tM} (C_{s^{(M)}}(A:\hat{B}) \cap t^{(M)}(\hat{B}))$.

Proof: By 3.19.

Corollary 3.21: Under the hypotheses of Corollary 3.20 the following are equivalent for a module Q :

- (i) Q is a direct summand in each extension N such that $N \subseteq Q + \sum_M r_{tM} (C_{s^{(M)}}(Q:\hat{N}) \cap t^{(M)}(\hat{N}))$,
- (ii) $Q \cong \sum_M r_{tM} (C_{s^{(M)}}(Q) \cap t^{(M)}(\hat{Q}))$,
- (iii) every diagram (1) with $M \subseteq N + \sum_U r_{tU} (C_{s^{(U)}}(N:\hat{M}) \cap t^{(U)}(\hat{M}))$ can be made commutative,
- (iv) Q is $(s, t, \text{zer}, \text{id})$ -injective.

Proof: By Corollary 3.20 and Theorem 1.3.

Corollary 3.22: Let $s^{(M)}$ and $t^{(M)}$ be as in 3.20. For any $Q \in R\text{-mod}$ define the sequence of modules Q_α inductively as follows:

$Q_0 = Q$, $Q_{\alpha+1} = Q_\alpha + \sum_M r_{tM} (C_{s^{(M)}}(Q_\alpha:\hat{Q}) \cap t^{(M)}(\hat{Q}))$ and $Q_\alpha = \bigcup_{\beta < \alpha} Q_\beta$, α limit. Then the module $\bar{Q} = Q_\alpha$ where $Q_\alpha = Q_{\alpha+1}$ is the smallest $(s, t, \text{zer}, \text{id})$ -injective submodule of \hat{Q} containing Q .

Proof: By Corollary 3.20 and Theorem 1.9.

Corollary 3.23: For every module M let $t^{(M)}$ be a

balanced preradical for $R\text{-mod}$ and $s^{(M)} = s$ be a hereditary radical. Then the module $\bar{Q} = Q + \sum_M r_{\{M\}}(C_s(Q:\hat{Q}) \cap t^{(M)}(\hat{Q}))$ is the smallest $(s, t, \text{zer}, \text{id})$ -injective submodule of \hat{Q} containing Q .

Proof: By Lemma 3.14 and Corollary 3.22.

R e f e r e n c e s

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Matematicko-fyzikální fakulta
 Karlova universita
 Sokolovská 83, 18600 Praha 8
 Československo

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