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# ON WEINGARTEN SURFACES

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*Abstract*: A global characterization of spheres.

*Key words*: Surface, sphere.

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Following the ideas of [1], I am going to prove the following

*Theorem*. *Let* $G$ *be a bounded domain in* $\mathbb{R}^2$ , $\partial G$ *its boundary and* $M: G \cup \partial G \longrightarrow E^3$ *a surface such that*: (i) $M(\partial G)$ *consists of umbilical points*; (ii) *there are functions* $g(x,y)$, $F(x,y)$ *and, on* $M$ , *orthonormal tangent vector fields* $v_1$, $v_2$ *satisfying*

$$(1) \qquad v_1 g(h,K) = 0 , \qquad v_2 f(h,K) = 0 ,$$

$H$ *and* $K$ *being the mean and Gauss curvatures of* $M$ *resp. Further, let*

$$(2) \qquad K > 0 , \qquad g_H g_K > 0 , \qquad f_H f_K > 0$$

*Then* $M(G \cup \partial G)$ *is a part of a sphere*.

*Proof*. On $M$ , consider a field of orthonormal moving frames $\{m; v_1, v_2, v_3\}$ . Then

$$(3)\qquad \begin{aligned} dm &= \omega^1 v_1 + \omega^2 v_2 , \\ dv_1 &= \qquad\quad \omega_1^2 v_2 + \omega_1^3 v_3 , \\ dv_2 &= -\omega_1^2 v_1 \qquad\quad + \omega_2^3 v_3 , \\ dv_3 &= -\omega_1^3 v_1 - \omega_2^3 v_2 \end{aligned}$$

with the usual integrability conditions. We have

$$(4)\qquad \omega_1^3 = a\,\omega^1 + b\,\omega^2 , \quad \omega_2^3 = b\,\omega^1 + c\,\omega^2 ,$$

$$(5)\qquad \begin{aligned} da - 2\,b\,\omega_1^2 &= \alpha\,\omega^1 + \beta\,\omega^2 \\ db + (a-c)\,\omega_1^2 &= \beta\,\omega^1 + \gamma\,\omega^2 \\ dc + 2\,b\,\omega_1^2 &= \gamma\,\omega^1 + \delta\,\omega^2 \end{aligned}$$

$$(6)\qquad \begin{aligned} dH &= \frac{1}{2}(\alpha+\gamma)\,\omega^1 + \frac{1}{2}(\beta+\delta)\,\omega^2 , \\ dK &= (a\gamma + c\alpha - 2b\beta)\,\omega^1 + (a\delta + c\beta - 2b\gamma)\,\omega^2 . \end{aligned}$$

On $M$, choose a curvilinear coordinate system $(u,v)$ such that

$$(7)\qquad I = r^2\,du^2 + s^2 dv^2 , \quad \omega^1 = r\,du, \quad \omega^2 = s\,dv, \quad rs \neq 0$$

Then

$$(8)\qquad \omega_1^2 = -\frac{r_v}{s}\,du + \frac{s_u}{r}\,dv .$$

From (5) and (7),

$$(9)\qquad \begin{aligned} d(a-c) &= 4\,b\,\omega_1^2 + (\alpha-\gamma)\,\omega^1 + (\beta-\delta)\,\omega^2 , \\ db &= -(a-c)\,\omega_1^2 + \beta\,\omega^1 + \gamma\,\omega^2 ; \end{aligned}$$

$$(10)\qquad (a-c)_u + 4\,b\,\frac{r_v}{s} = (\alpha-\gamma)\,r\ ,$$

$$b_u \qquad - (a-c)\,\frac{r_v}{s} = \beta r\ ,$$

$$(a-c)_v - 4\,b\,\frac{s_u}{r} = (\beta-\delta)\,s\ ,$$

$$b_v \qquad + (a-c)\,\frac{s_u}{r} = \gamma\, s$$

and

$$(11)\qquad \alpha\, rs = s(a-c)_u + rb_v + (.)\,(a-c) + (.)b\ ,$$

$$\beta\, rs = s\, b_u + (.)\,(a-c) + (.)b\ ,$$

$$\gamma\, rs = r\, b_v + (.)\,(a-c) + (.)b\ ,$$

$$\delta\, rs = -\,r(a-c)_v + s\, b_u + (.)\,(a-c) + (.)b\ .$$

From (1),

$$(12)\qquad g_H\, dH(v_1) + g_K\, dK(v_1) = 0\ ,$$

$$f_H\, dH(v_2) + f_K\, dK(v_2) = C\ ,$$

i.e.,

$$(13)\qquad g_H(\alpha+\gamma) + 2\, g_K(a\gamma + c\alpha - 2b\beta) = 0\ ,$$

$$f_H(\beta+\delta) + 2\, f_K(a\delta + c\beta - 2b\gamma) = 0\ .$$

Because of (11), (13) turns out to be

$$(14)\qquad \{ sg_H - 2\, cs\, g_K \}\, (a-c)_u - 4\, bs\, g_K\, b_u +$$

$$+ \{ 2\, rg_H + 2\, rg_K(a+c) \}\, b_v = (.)(a-c) + (.)b,$$

$$\{-rf_H - 2\,ar\,f_K\}(a-c)_v + \{2\,sf_H + 2\,sf_K(a + c)\}\,b_u - 4\,br\,f_K\,b_v = (.)\,(a-c) + (.)b\ .$$

This system can be written in the form

$$(15)\quad a_{11}(a-c)_u + a_{12}(a-c)_v + b_{11}b_u + b_{12}b_v = c_{11}(a-c) + c_{12}b\ ,$$

$$a_{21}(a-c)_u + a_{22}(a-c)_v + b_{21}b_u + b_{22}b_v = c_{21}(a-c) + c_{22}b$$

with

$$(16)\quad a_{11} = s(g_H + 2\,ag_K), \qquad a_{12} = 0\ ,$$

$$b_{11} = -4\,b\,sg_K\ , \qquad b_{12} = 2\,r(g_H + 2\,H\,g_K)\ ,$$

$$a_{21} = 0\ , \qquad a_{22} = -r(f_H + 2\,a\,f_K)\ ,$$

$$b_{21} = 2\,s(f_H + 2\,H\,f_K)\ , \quad b_{22} = -4\,br\,f_K$$

The system (15) is called elliptic if the form

$$(17)\quad \Phi = (a_{12}b_{22} - a_{22}b_{12})\mu^2 - (a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11})\mu\nu + (a_{11}b_{21} - a_{21}b_{11})\nu^2$$

is definite. From (16),

$$a_{12}b_{22} - a_{22}b_{12} = 2\,r^2(g_H + 2\,H\,g_K)(f_H + 2\,a\,f_K)\ ,$$

$$a_{11}b_{22} - a_{21}b_{12} + a_{12}b_{21} - a_{22}b_{11} = -4\,b\,sr\,\{f_K(g_H +$$

$$+ 2\ c\ g_K) + g_K(f_H + 2\ a\ f_K)\} ,$$

$$a_{11}b_{21} - a_{21}b_{11} = 2\ s^2(g_H + 2\ c\ g_K)(f_H + 2\ H\ f_K) .$$

The discriminant of $\Phi$ being denoted by $\Delta$ , we get

$$(18) \qquad \Delta = 16\ r^2 s^2 [2\ H\ f_H f_K g_H^2 + 2\ H\ g_H g_K f_H^2 + 2(\tfrac{1}{2} f_H g_H + c\ f_H g_K)^2 + 2(\tfrac{1}{2} f_H g_H + a\ f_K g_H)^2 + (2\ K + b^2)(f_H^2 g_K^2 + f_K^2 g_H^2) + (12\ H^2 + 4\ K + 2\ b^2)\ f_H\ f_K\ g_H\ g_K + 8\ H(b^2 + c^2 + 2\ K)\ f_H\ f_K\ g_K^2 + 8\ H(a^2 + b^2 + 2\ K)\ g_H\ g_K\ f_K^2 + 16\ H^2\ K\ f_K^2\ g_K^2] .$$

From (2), $\Delta > 0$ , and the system (14) is elliptic. On $\partial G$ , $a - c = b = 0$ ; the ellipticity of (14) implies $a - c = b = 0$ on $G$ . Because of $4(H^2 - K) = (a - c)^2 + 4\ b^2 = 0$ , $M$ is a part of a sphere. QED.

The H- and K-theorems are now trivial consequences of our Theorem.

## Reference


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