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BANACH SPACES ON WHICH EVERY UNCONDITIONALLY CONVERGING OPERATOR IS COMPLETELY CONTINUOUS

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Abstract: For continuous linear operators acting between Banach spaces, it is known that if an operator T sends weak Cauchy sequences into norm convergenct sequences, then T sends weakly unconditionally convergent (wuc) series into unconditionally convergent (uc) series. This paper is devoted to study Banach spaces with the property that for every linear operator T from such a space to an arbitrary Banach space the converse implication is true, i.e. if T sends every wuc series into an uc series, then T sends weak Cauchy sequences into norm convergent sequences. $C(S)$, c_0, and ℓ_∞ are among the Banach spaces which have this property while infinite dimensional reflexive spaces are among those that do not.

Key words and phrases: Linear operator, weak Cauchy sequence, unconditionally convergent series, limited set.

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Let X and Y be Banach spaces and T be a bounded linear operator from X to Y. T is said to be <u>completely continuous</u> if T maps weak Cauchy sequences in X into norm convergent sequences in Y. T is said to be <u>unconditionally converging</u> (<u>uc operator</u>) if it maps weakly unconditionally convergent (wuc) series in X into unconditionally convergent (uc) series in Y. Recall that a series is wuc iff $\sum|x'(x_n)| < \infty$ for each $x' \in X'$, and the series $\sum x_n$ is uc iff every rearrangement converges in the norm topology of X.

Conditions for an operator to be completely continuous or uc in terms of limited sets are given in Section 1 while Section 2 is about B-spaces on which these two operators are equivalent.

1. Limited sets and linear operators. Let X denote a Banach space and X' its conjugate. Then for subsets A of X and A' of X', we define conditions for these subsets to be limited.

Definition 1.1. (a) A' is wuc-limited if $\lim_n \sup_A x'(x_n) = 0$ for every wuc series $\sum x_n$ in X.

(b) A is $\widetilde{\text{wuc}}$-limited if $\lim_n \sup_A x_n'(x) = 0$ for every wuc series $\sum x_n'$ in X'.

(c) A' is w-limited if $\lim_n \sup_{A'} |x'(x_n)| = 0$ for every sequence $\{x_n\}$ which converges weakly to 0 in X.

(d) A is $\tilde{w}$-limited if $\lim_n \sup_A |x_n'(x)| = 0$ for every sequence $\{x_n'\}$ which converges weakly to 0 in X'.

Note that if A' is $\tilde{w}$-limited, then A' is w-limited and if A' is $\widetilde{\text{wuc}}$-limited, then A' is wuc-limited. Furthermore, we have:

Proposition 1.2. (a) A' w-limited implies A' wuc-limited, and

(b) A $\tilde{w}$-limited implies A $\widetilde{\text{wuc}}$-limited.

In general, the converses of Proposition 1.2 are not true. Actually Section 2 is about those Banach spaces for which these converses are true. To analyze that, we first characterize the completely continuous and uc operators in terms of the above types of limited sets. For a linear operator T mapping a Banach space X into a Banach space Y, T' $(T': Y' \longrightarrow X')$ is to denote the adjoint of T.

__Proposition 1.3.__ Suppose $T: X \longrightarrow Y$ and $T': Y \longrightarrow$ $\longrightarrow X'$.

(a) T is completely continuous iff T' maps bounded sets into w-limited sets.

(b) T' is completely continuous iff T maps bounded sets into $\tilde{w}$-limited sets.

__Proof of (a)__: Let T be completely continuous and $\{x_n\} \xrightarrow{w} 0$. Then $\lim \| Tx_n \| = 0$. Hence

$$\lim_n \sup_{A'} | y'(Tx_n) | = \lim_n \sup_{T'A'} | T'y'(x_n) | = 0 .$$

Therefore $T'A'$ is w-limited. The converse is the reverse argument.

__Proposition 1.4.__ Suppose $T: X \longrightarrow Y$ and $T': Y' \longrightarrow$ $\longrightarrow X'$.

(a) T is uc iff T' maps bounded sets into wuc-limited sets.

(b) T' is uc iff T maps bounded sets into $\widetilde{wuc}$-limited sets.

__Proof of (a)__: Let T be uc and $\sum x_n$ be wuc. Then $\sum Tx_n$ is uc. Hence $Tx_n \longrightarrow 0$ and we obtain

$$\lim_n \sup_{A'} y'(tx_n) = \lim_n \sup_{T'A'} T'y'(x_n) = 0$$

Therefore $T'A'$ is wuc-limited.

Now suppose $\sum x_n$ is wuc and $\lim_n \sup_{T'A'} T'y'(x_n) = 0$.

Then $\lim_n \sup_{A'} y'(Tx_n) = 0$. By taking A' to be the unit cell of X', we have $\lim_n \| Tx_n \| = 0$, so $\inf_n \| Tx_n \| = 0$. Now by Theorem 5 of [1], $\sum Tx_n$ is uc.

Since a w-limited set is wuc-limited, it follows that if $T: X \longrightarrow Y$ is completely continuous, then T is uc. Another interesting consequence is:

Corollary 1.5. Let v and v' be the unit cells of X and X' respectively.

(a) v' is wuc-limited iff no subspace of X is isomorphic to c_0.

(b) v is $\widetilde{\text{wuc}}$-limited iff no subspace of X' is isomorphic to c_0.

Proof of (a): v' is wuc-limited iff wuc series and uc series coincide in X. The result follows by Theorem 5 of [1].

2. P(uc,cc) and $\widetilde{P}$(uc,cc) properties. We denote the space of bounded scalar-valued functions on an arbitrary topological space S with the sup norm by $B(S)$. An operator representation theorem for completely continuous and uc operators is given as follows:

Lemma 2.1. Let $T: X \longrightarrow B(S)$ and $p: S \longrightarrow X'$ be such that p is a bounded map and $Tx(s) = p(s)x$.

(a) T is completely continuous iff $p(S)$ is w-limited.

(b) T is uc iff $p(S)$ is wuc-limited.

<u>Proof of (a)</u>: Assume T is completely continuous and let $\{x_n\} \xrightarrow{w} 0$. Then $\|Tx_n\| \longrightarrow 0$; hence $\lim_n \sup_S |Tx_n(s)| = \lim_n \sup_{p(S)} |p(s)x| = 0$.

Therefore, $p(S)$ is w-limited.

Conversely, let S be any bounded subset of X' and assume that $p(S)$ is w-limited. Let $\{x_n\} \xrightarrow{w} 0$. Then

$$\lim_n \sup_{p(S)} |p(s)x_n| = \lim_n \sup_S |Tx_n(s)| = 0$$

Thus $\|Tx_n\| \longrightarrow 0$, so T is completely continuous,

<u>Proposition 2.2</u>. Let X be a B-space. Then the following conditions are equivalent:

(a) For every B-space Y , every uc operator $T: X \longrightarrow Y$ is completely continuous.

(b) Every wuc-limited subset of X' is w-limited.

<u>Proof</u>: (a) $\Longrightarrow$ (b): Let A' be a wuc-limited set in X' and $p = i$, the identity, mapping A' into X' in the hypothesis of Lemma 2.1. Then T is uc, hence completely continuous. Again by Lemma 2.1, A' is w-limited.

(b) $\Longrightarrow$ (a): This follows from Propositions 1.3 and 1.4.

<u>Definition 2.3.</u> A Banach space is said to have the P(uc,cc) <u>property</u> if it satisfies one of the equivalent contidions (a) or (b) of Proposition 2.2.

Let X be a B-space. If for every B-space Y, every uc operator $T: X \longrightarrow Y$ is weakly compact, X is said to have <u>property</u> V [3]. If for every B-space Y, every weakly compact operator $T: X \longrightarrow Y$ is completely continuous, X is said to have the <u>Dunford-Pettis (DP)</u> property. Hence, if a space X has both the V and DP properties then X will have the P(uc,cc) property. Banach spaces which satisfy the V and DP properties are $C(S)$ (S compact Hausdorff), c_0, and ℓ_∞.

<u>Proposition 2.4</u>. The following are equivalent:

(a) Weak and norm convergence of sequences coincide in X.

(b) X is weakly sequentially complete and has the P(uc,cc) property.

<u>Proof</u>: (a) $\Longrightarrow$ (b): Clearly X is weakly sequentially complete. Since every operator on X will be completely continuous, X will have the P(uc,cc) property.

(b) $\Longrightarrow$ (a): Since X is weakly sequentially complete, no subspace of X is isomorphic to c_0 (see [1]). Then by Corollary 1.5 (a), v' is wuc-limited and hence w-limited. But this is equivalent to (a).

Since the only weakly complete B-spaces which have the P(uc,cc) property are those in which weak and norm convergence of sequences coincide, ℓ_1 has the P(uc,cc) property while ℓ_∞ does not. Also, an infinite dimensional reflexive B-space cannot have the P(uc,cc) property. If X is a B-space having the P(uc,cc) property, then every quotient space of X has the P(uc, cc) property. The Cartesian product $X_1 \times X_2$ has the P(uc,cc) property if and only if X_1 and X_2 have the P(uc,cc) property.

Definition 2.5. X is said to have the $\tilde{P}$(uc,cc) property if every $\widetilde{\text{wuc}}$-limited set of X is also $\tilde{w}$-limited.

The following results are similar to those for the P(uc,cc) property.

Proposition 2.6. (a) X has $\tilde{P}$(uc,cc) iff for every B-space Y, every uc operator $T': X' \longrightarrow Y'$ is completely continuous.

(b) Weak and norm convergence of sequences coincide in X' iff X' is weakly complete and X has the $\tilde{P}$(uc,cc) property.

As a consequence of this proposision c_0 has the $\tilde{P}$(uc,cc) . Note that ℓ_∞ has P(uc,cc) , but not $\tilde{P}$(uc,cc) while ℓ_∞ has $\tilde{P}$(uc,cc) , but not P(uc,cc) .

Proposition 2.7. If X' has the P(uc,cc) property, then X has the $\tilde{P}$(uc,cc) property.

Proof: Let A be $\widetilde{\text{wuc}}$-limited. Then $\hat{A}$ is wuc-limited in X" and hence $\hat{A}$ is w-limited since X has the P(uc,cc) property. Thus A is $\tilde{w}$-limited.

The converse of Proposition 2.7 is not true. Stegall [4] givas and example of a space E in which weak and norm convergence of sequences coindie while E' has a complemented (infinite dimensional) subspace which is reflexive. So E has $\tilde{P}$(uc,cc) , but E' cannot have the P(uc,cc) property.

Pelczynski [3] defines a B-space X to have the property V* if every $\widetilde{\text{wuc}}$-limited set in X is weakly sequentially compact. An equivalent condition for X to have the DP property is that every weakly sequentially compact set of X is $\widetilde{w}$-limited [2]. Hence if X has both properties V* and DP, it has the $\widetilde{P}$(uc,cc) property. Consequently, we have the following proposition.

Proposition 2.8. Every abstract L-space has the $\widetilde{P}$(uc,cc) property.

Contained in the proof of Proposition 6 of [3] is the following:

Lemma 2.9. (a) A weak Cauchy sequence in X is a $\widetilde{\text{wuc}}$-limited set.

(b) A weak Cauchy sequence in X' is a wuc-limited set.

This lemma and Proposition 1.4 will give the fact that every weakly compact operator is uc [3]. Hence we have the following result.

Proposition 2.10. If X has either the P(uc,cc) or $\widetilde{P}$(uc,cc) property, then X has the DP property.

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