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PARTITIONS OF VERTICES

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DEDICATED TO EMANUEL GRINBERG ON THE OCCASION OF HIS 65-TH
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Abstract: In this note we show how a theorem by Erdős-Hajnal may be used for proving theorems concerned with partitions of vertices of graphs, relations etc.

Key-words: Ramsey theorem, partitions, chromatic number.

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Introduction. In 1966 Erdős and Hajnal [1] proved the following.

Theorem A : For every positive integer $k \geq 2$, $\ell \geq 2$, $n \geq 1$ there exists a hypergraph $\mathcal{G} = \mathcal{G}(k, \ell, n) = (X, \mathcal{M})$ with the following properties:

- 1) \mathcal{G} is a k -uniform hypergraph
- 2) \mathcal{G} does not contain cycles of length smaller than ℓ
- 3) $\chi(\mathcal{G}) > n$

The notation is the following: $\chi(\mathcal{G})$ = chromatic number of \mathcal{G} i.e. the minimal number of colours which are necessary for colouring the vertices of \mathcal{G} in such a way that no monocoloured hyperedge occurs; k -uniform means that $|M| = k$ for every $M \in \mathcal{M}$; a cycle of length ℓ is a sequence $x_1, M_1, x_2, M_2, \dots, x_\ell, M_\ell$ such that $x_i \in M_i$, $i \in [1, \ell]$;

$x_{i+1} \in M_i, i \in [1, \ell - 1]; x_1 \in M_\ell$

$\{M_i \mid i \in [1, \ell]\} \subseteq \mathcal{M}, \{x_i \mid i \in [1, \ell]\} \subseteq X.$

To avoid the trivial cycle consisting of only one hyperedge we assume that there are i, j such that $M_i \not\subseteq M_j$.

Theorem A was proved by nonconstructive means. In 1968 L. Lovász proved the same theorem constructively.

In this note we show how this theorem implies (using a simple trick) a very general theorem of Ramsey type for partitions of vertices. There are two reasons for publishing of this note: first, the trick provides simpler proofs to known theorems ([2],[3],[4]), secondly, partitions of vertices are used as a tool for proving a Ramsey type theorem for partitions of edges and we shall need a general theorem for partitions of vertices for our forthcoming papers.

We apply the Theorem A to partitions of vertices of graphs, hypergraphs, relations and universal algebras. In § 4 we show that given a graph G there exists an infinite set of minimal graphs with the vertex partition property for G . We end this note with a few problems and comments concerning infinite graphs.

1. Folkman's theorem. In 1967 J. Folkman [3] proved: For every positive integer r and for every graph $G = (V, E)$ without complete subgraphs on m vertices there exists a graph $H = (W, F)$ without complete subgraphs on m vertices such that for every partition $W = \bigcup_{i=1}^r W_i$ there exists an i and an embedding $f: G \rightarrow H$ such that $f(V) \subseteq W_i$ (An embedding $f: G \rightarrow H$ is an 1-1 mapping with the property

$\{f(x), f(y)\} \in F \iff \{x, y\} \in E$.). We denote by $G \xrightarrow[r]{v} H$ the validity of the above statement for G, H , the negation is denoted by $G \not\xrightarrow[r]{v} H$. This notation has the following sense: Let $G \rightarrow H$ denote the fact that there exists an embedding of G into H . Then $G \xrightarrow[r]{v} H$ means that there are "so many" embeddings of G into H that even if we partition vertices of H into r parts we still have an embedding in one of the parts. In this way $\xrightarrow[r]{v}$ may be seen as a combinatorially strengthened embedding arrow (see [8]).

Folkman gave a direct constructive proof of the above fact. Another (less elementary) proof is due to the authors of [7]. However, Theorem A instantly yields a much stronger result.

Definition: Let K be a fixed graph. Denote by $\text{Gra}(K)$ the class of all graphs which do not contain K as a subgraph. (I.e. $G \notin \text{Gra}(K) \iff$ there are sets $V_0 \subseteq V, E_0 \subseteq E$ such that $(V_0, E_0) \cong K$.)

If \mathcal{K} is a set of graphs put $\text{Gra}(\mathcal{K}) = \bigcap \{ \text{Gra}(K) \mid K \in \mathcal{K} \}$.

Theorem 1: Let \mathcal{K} be a finite set of 2-connected graphs. Then for every graph $G \in \text{Gra}(\mathcal{K})$ there exists a graph $H \in \text{Gra}(\mathcal{K})$ such that $G \xrightarrow[r]{v} H$.

We may assume $|K| > 2$ for every $K \in \mathcal{K}$ as for $|K| \leq 2$ we get either the void class of graphs or the class of all discrete graphs.

Proof: Let $G = (V, E) \in \text{Gra}(\mathcal{K})$ be fixed. Let $b = \max_{K \in \mathcal{K}} |K| + 1, |G| = k$. Let us choose $\mathcal{G}(k, \ell, r) = (X, \mathcal{M})$

with the properties of Theorem A. For each $M \in \mathcal{M}$ let $f_M: V \rightarrow M$ be a fixed bijection. Define the graph $H = (X, F)$ such that $\{x, y\} \in F \iff$ there exist $M \in \mathcal{M}$ and $\{z, t\} \in E$ such that $\{f_M(z), f_M(t)\} = \{x, y\}$. This graph H will be denoted by $(X, \mathcal{M}) * G$.

As $\ell > 2$ we have $|M \cap N| \leq 1$ whenever $M \neq N$, $\{M, N\} \subseteq \mathcal{M}$ (see above) and consequently

1) $f_M: V \rightarrow M$ is an embedding of G into H for each $M \in \mathcal{M}$.

2) If K' is a subgraph of H , $K' \cong K \in \mathcal{K}$ then $K' \subseteq (M, F)$, $M \in \mathcal{M}$.

(This follows by the 2-connectivity of K and by the fact that (X, \mathcal{M}) does not contain a cycle of length $< |K| + 1$.)

Finally $G \xrightarrow[\kappa]{\nu} H$ follows immediately from $\chi(X, \mathcal{M}) > r$:

Given a partition $X = \bigcup_{i=1}^r X_i$ there exists $M \in \mathcal{M}$ and $i \in [1, r]$ such that $M \subseteq X_i$. Consequently G is an induced subgraph of (X_i, F) and f_M is an embedding.

This theorem does not hold for graphs with connectivity < 2 :

i) If K is disconnected and $K = K' \cup K''$ where $K' \cong K''$ then $H \not\xrightarrow[\kappa]{\nu} K'$ for every $H \in \text{Gra}(K)$ as may be seen easily.

ii) If $K = P_n$ is a path of length n then $G \in \text{Gra}(P_n) \implies \implies \chi(G) \leq n$. From this follows that there exists $G \in \text{Gra}(P_n)$ such that $G \not\xrightarrow[\kappa]{\nu} H$ for every graph $H \in \text{Gra}(P_n)$ (it suffices to take $G \in \text{Gra}(P_n)$ which satisfies $\chi(G) > \frac{n}{2}$; obviously $G \xrightarrow{\nu} H \implies \chi(H) \geq$

$\geq 2 \chi(G) - 1$).

iii) If \mathcal{X} is an infinite set then the statement may be false (consider $\mathcal{X} = \{C_{2k+1} \mid k \geq 1\}$ the set of all odd cycles).

2. Partitions of vertices of relations and hypergraphs. Using the same ideas as in 1 we may prove analogous theorems for relations and hypergraphs. We list only statements:

Theorem 2a: Let \mathcal{R} be a finite set of 2-weakly connected relations (see [5], p.199). Then for every positive integer r and for every $R \in \text{Rel}(\mathcal{R})$ there exists $S \in \text{Rel}(\mathcal{R})$ such that $R \xrightarrow[r]{v} S$.

Theorem 2b: Let \mathcal{F} be a finite set of hypergraphs which are 2-connected (i.e. (X, \mathcal{M}) is 2-connected $\iff \iff (X, \cup\{P_r(M) \mid M \in \mathcal{M}\})$ is a 2-connected graph). Then for positive integer r and for every $(X, \mathcal{M}) \in \text{Hyp}(\mathcal{F})$ there exists $(Y, \mathcal{N}) \in \text{Hyp}(\mathcal{F})$ such that

$$(X, \mathcal{M}) \xrightarrow[r]{v} (Y, \mathcal{N}).$$

The definitions of $\text{Rel}(\mathcal{R})$ and $\text{Hyp}(\mathcal{F})$ and of symbols

$\xrightarrow[r]{v}$ are quite analogous to the definitions $\text{Gra}(\mathcal{X})$ and $\xrightarrow[r]{v}$ for graphs. Again in certain sense these theorems are best possible.

3. Partitions of algebras. Let \mathcal{V} be the class of all finite universal algebras of given type $\Delta = (n_i \mid i \in I)$.

Let $\mathfrak{X} = (X, (\omega_i \mid i \in I))$, $\mathfrak{Y} = (Y, (\alpha_i \mid i \in I))$ be algebras from \mathcal{V} . We write $\mathfrak{X} \xrightarrow{x} \mathfrak{Y}$ iff for every partition $Y = \bigcup_{i=1}^n Y_i$ there exists an i and a monomorphism $f: \mathfrak{X} \rightarrow \mathfrak{Y}$ such that $f(x) \in Y_i$.

Theorem 3: Let r be positive integer and let $\mathfrak{X} \in \mathcal{V}$ and $\omega_i(x, x, \dots, x) = x$ for every $i \in I$ and $x \in X$ (idempotent algebras). Then there exists $\mathfrak{Y} \in \mathcal{V}$ such that $\mathfrak{X} \xrightarrow{x} \mathfrak{Y}$.

Proof: Let $|X| = k$, $l = 3$, $n = r$. Consider $(Y', \mathcal{M}) = \mathcal{G}(k, 3, r)$. Let $y' \in Y'$. For every $M \in \mathcal{M}$ let us choose a bijection $f_M: X \rightarrow M$. Define $(Y, (\alpha_i \mid i \in I))$ by $Y = Y' \cup \{y'\}$ and $\alpha_i(y_j \mid j \in [1, n_i]) = f_M(\omega_i(x_j \mid j \in [1, n_i]))$ where $f_M(x_j) = y_j$ if such an M exists, otherwise we put $\alpha_i(y_j \mid j \in [1, n_i]) = y'$, $i \in I$. It is easy to check that $\mathfrak{Y} \in \mathcal{V}$, $\mathfrak{X} \xrightarrow{x} \mathfrak{Y}$.

Again it is easy to see that, generally, for non-idempotent algebras Theorem 3 fails to be true.

Remark: A very difficult problem seems to be the characterization of those primitive classes of algebras for which the statement analogous to Theorem 3 holds. This is true for example for the class of all finite distributive lattices.

4. **Critical Folkman graphs:** Let G be a graph. We say that H is an irreducible (r, v) -graph for G if $G \xrightarrow{v} H$ but $G \not\xrightarrow{v} H'$ for every proper subgraph

H' of H .

Theorem 4 a: For every graph G , $|G| > 1$ there exists a countable set of non-isomorphic irreducible (r,v) -graphs for G .

Proof: A proof follows directly from the $*$ construction in 1. Let G be fixed. We may assume that G is a connected graph (otherwise we consider the complement of G) .

It suffices to put

$$\begin{aligned} H_1 &= \mathcal{F}(|G|, 3, r) * G \\ H_2 &= \mathcal{F}(|G|, |H_1|, r) * G \\ &\vdots \\ &\vdots \\ H_{n+1} &= \mathcal{F}(|G|, |H_n|, r) * G \\ &\vdots \\ &\vdots \end{aligned}$$

$G \xrightarrow[\kappa]{\nu} H_i$ holds for every i . Let \bar{H}_i be an irreducible (r,v) -graph for G contained in H_i , $i = 1, 2, \dots$. Obviously $|H_i| < |H_j|$ for all i, j satisfying $i < j$. Assume $\bar{H}_i \cong \bar{H}_j$ for $i < j$. As $\bar{H}_j \subseteq H_j$, $H_j = \mathcal{F}(|G|, |H_{j-1}|, r) * G$, and $|H_{j-1}| \geq |H_i|$ we have $|\bar{H}_i| \subseteq \bar{\mathcal{F}} * G$ where $\bar{\mathcal{F}} \subseteq \mathcal{F}(|G|, |H_{j-1}|, r)$ is a hypergraph which does not contain any cycle. But in this case $\chi(\bar{\mathcal{F}} * G) = \chi(G)$. This contradicts

$$G \xrightarrow[\kappa]{\nu} \bar{H}_i .$$

Remark 1: Using a modified proof we may even prove

Theorem 4 b: Let \mathcal{K} be a finite set of 2-connected graphs. Then for every graph $G \in \text{Gra}(\mathcal{K})$ there exists a countable set $\{H_i \mid i = 1, 2, \dots\}$ of non-isomorphic graphs such that

- 1) $H_i \in \text{Gra}(\mathcal{K})$
- 2) H_i is an irreducible (r, v) -graph for G .

Remark 2: Theorem 4a does not hold for infinite graphs. Every complete graph of infinite cardinality is the only (r, v) -irreducible graph for itself. Theorem 4a fails to be true for $|G| = 1$, too.

Remark 3: Let $G = (V, E)$, $H = (W, F)$ be graphs. We write $G \xrightarrow[r]{e} H$ if for every partition $F = \bigcup_{i=1}^r F_i$ there exists an embedding $f: G \rightarrow H$ such that $\{\{f(x), f(y)\} \mid \{x, y\} \in E\} \subseteq F_i$ for an $i \in [1, r]$. The existence of an Ramsey graph for every finite graph was proved independently by Deuber, Erdős, Hajnal, Posa and Rödl [9], see also much stronger [7].

Define H to be an (r, e) -irreducible graph for G if $G \xrightarrow[r]{e} H$ but $G \not\xrightarrow[r]{e} H'$ for all proper subgraphs H' of H .

Problem 1: Characterize those finite graphs G for which there exists an infinite set of non-isomorphic (r, e) -irreducible graphs for G .

If a graph G contains at most one edge then there exists precisely one (r, e) -irreducible graph H such that

$G \xrightarrow[r]{e} H$, namely G itself.

Conjecture: For a finite graph G the following two statements are equivalent:

1. For G there exists a countable set of (r, e) -irreducible graphs.
2. $G \xrightarrow[\kappa]{e} G$.

The path of length 2 is an example of a graph G for which there exists a countable set of $(2, e)$ -irreducible graphs.

One can take the family of all odd cycles.

More generally, the same is true for every paths of length l , l finite.

Finally let us remark that Theorem 1 shows the power of Erdős-Hajnal theorem for partitions of vertices.

There is no general method known for deriving similar theorems for partitions of edges (see [8] for results in this direction). Let us add a few remarks concerning infinite graphs. In an obvious way we may extend the symbol $G \xrightarrow[\kappa]{r} H$ for infinite graphs G , H and any cardinal r . The following is then true:

Theorem 5a: For every graph G and every positive integer r there exists H such that $G \xrightarrow[\kappa]{r} H$.

Theorem 5b: For every finite graph G and every cardinal r there exists a graph H such that $G \xrightarrow[\kappa]{r} H$. Moreover, if G does not contain a complete graph on m vertices then H may be chosen with the same property.

Theorem 5a may be proved by the following construction:

Let $G = (V, E)$, assume without loss of generality $r = 2$

(this is possible as $G \xrightarrow{\frac{r}{2}} H \xrightarrow{\frac{r}{2}} I \implies G \xrightarrow{\frac{r}{4}} I$).

Put $H = (V \times V, F)$ where $\{(x,y), (x',y')\} \in F \iff$ either $x = x'$ $\{y,y'\} \in E$ or $\{x,x'\} \in E$.

Given a colouring $c: V \times V \rightarrow \{1,2\}$ either there exists

$x \in V$ such that $c(\{x\} \times V) = i$ or there exists i such that for every $x \in V$ there exists y with $c((x,y)) = i$.

From this follows easily $G \xrightarrow{\frac{r}{2}} H$.

Theorem 5b follows from the Erdős-Rado generalization of the classical Ramsey theorem for cardinal numbers and from the representation of finite graphs by type-graphs, see [7],[8]. This is a straightforward application of type-graphs and we omit the proof.

This leads to the following problems (see also [2]):

Problem 2: Let G be a graph and r a cardinal number. Does there exist a graph H such that $G \xrightarrow{\frac{r}{2}} H$?

Moreover, providing that G does not contain a complete graph with m vertices is it possible to choose H with the same property?

Not much is known, even the case $m = 3$ and $r = 2$ is unsolved. The purpose of this remark is to show that even dealing with vertex partitions one cannot be overoptimistic.

R e f e r e n c e s :

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