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EXACTNESS OF THE SET-VALUED COLIM

J. ADÁMEK, J. REITERMAN, PRAHA

Abstract: It is well-known that, in the category of sets, filtered colimits commute with finite limits; thus, if K is a filtered small category then the functor $\text{colim}: \text{Set}^K \rightarrow \text{Set}$ is exact (i.e. preserves regular epis and finite limits). The converse is proved in the present note and other properties of colim are investigated and compared with those of $\text{colim}: \text{Ab}^K \rightarrow \text{Ab}$ for the category Ab of Abelian groups.

Key words: Exact colimits, category of sets.

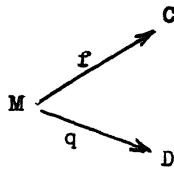
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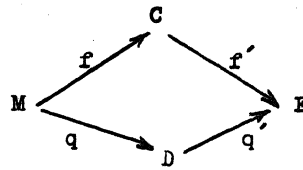
I. Formulation

I.1. The exactness of colim for Ab has been investigated by Isbell and Mitchell [2], [3]. In that case colim is exact iff it preserves equalizers and iff it preserves monics. For the set-valued colim (i.e. for $\text{colim}: \text{Set}^K \rightarrow \text{Set}$) these properties differ. We shall prove namely the following propositions (see part III).

I.2. (a) colim preserves monics iff every diagram (*)



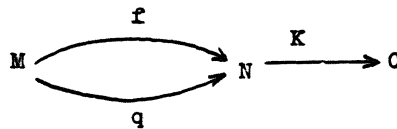
(*)



(**)

in K is a part of commutative square (**)

(b) colim preserves equalizers iff K has filtered components, i.e. iff K fulfils the condition of (a) and for every pair f, g of parallel morphisms there is k with $kf = kg$,



(c) colim is exact iff K is filtered, i.e. iff K fulfils the conditions of (a), (b) and for every pair A, B of K -objects there is C with $\text{Hom}(A, C) \neq \emptyset \neq \text{Hom}(B, C)$.

I.3. This characterization is rather simple in comparison with the Ab case. $\text{Colim}: \text{Ab}^K \rightarrow \text{Ab}$ is exact iff the following category $\text{aff } K$ has filtered components: objects of $\text{aff } K$ are just the objects of K ; morphisms from A to B are those elements $\sum \alpha_i f_i$ of the free Abelian group over $\text{Hom}_K(A, B)$ for which $\sum \alpha_i = 1$, see [3].

I.4. It is easily seen that 1) $\text{aff } K$ has filtered components provided that K has, 2) if $\text{aff } K$ has filtered components then K fulfils the condition of (a). Thus,

denoting $A = \text{colim} : \text{Ab}^K \rightarrow \text{Ab}$, $S = \text{colim} : \text{Set}^K \rightarrow \text{Set}$
 we get

S is exact $\implies S$ preserves equalizers $\implies A$ is exact
 $\implies S$ preserves monics

None of these implications can be reversed. The counterexamples are easy (according to I.2, I.3) except that to the second implication: for the category K of finite ordinals and order preserving injections, A is proved to be exact in [3] but the only component of K is not filtered.

II. Relation to indecomposable functors

II.1. Colimits in sets are closely related to indecomposability: a functor $F: K \rightarrow \text{Set}$ is indecomposable if whenever $F = F_1 \vee F_2$ then F_1 or F_2 is the constant functor to \emptyset . Notice that F is indecomposable iff $\text{colim } F$ is a singleton set.

Let us observe that each non-trivial functor $F: K \rightarrow \text{Set}$ can be decomposed into a sum of its components, i.e. maximal indecomposable subfunctors, $F = \coprod_{i \in I} F_i$. If $\mu: F \rightarrow F'$ is a transformation and $F' = \coprod_{j \in J} F'_j$ is a decomposition of F' into components then for every $i \in I$ there is $c(i) \in J$ with $\mu(F_i) \subset F'_{c(i)}$. We have $\text{colim } F = I$, $\text{colim } F' = J$, $\text{colim } \mu = c$. From these observations one can derive the following properties of $\text{colim}: \text{Set}^K \rightarrow \text{Set}$.

II.2. (a) colim preserves monics iff each non-trivial subfunctor of an indecomposable functor $F: K \rightarrow \text{Set}$ is indecomposable, too.

(b) colim preserves equalizers iff indecomposable

functors from K to Set have always the following "agreement property": for each couple $\mu, \nu : F \rightarrow F'$ of transformations there is M and $x \in FM$ with $(\mu_M x = \nu_M x)$.

(c) colim preserves finite products iff the product of two indecomposable functors from K to Set is indecomposable, too.

II.3. The exactness of colim in the Ab case can be also characterized analogously [1]: $\text{colim}: \text{Ab}^K \rightarrow \text{Ab}$ is exact iff the agreement property from (b) holds for all couples of endo-transformations of indecomposable functors from K to Set ; equivalently, iff each endotransformation $\mu : F \rightarrow F$ of an indecomposable functor $F: K \rightarrow \text{Set}$ has a fixed point (i.e. x in some FM with $(\mu_M x = x)$).

III. Proof

III.1. Necessities in I.2 follow from II.2 if we take into account that

(a) the subfunctor F of $\text{Hom}(M, -)$ generated by $f : M \rightarrow C$, $g : M \rightarrow D$ must be indecomposable (then we have $f' : C \rightarrow E$, $g' : D \rightarrow E$ with $f'f = g'g$),

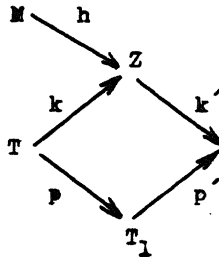
(b) the transformations $\text{Hom}(f, -)$, $\text{Hom}(g, -) : \text{Hom}(N, -) \rightarrow \text{Hom}(M, -)$ must coincide at some $k \in \text{Hom}(N, C)$; and all monics are equalizers in Set^K ,

(c) the product $\text{Hom}(M, -) \times \text{Hom}(N, -)$ must be non-trivial.

III.2. Sufficiencies. (a) Let $f : K \rightarrow \text{Set}$ be an indecomposable functor. To prove that all subfunctors of F

are indecomposable it suffices, for given $x \in FM$, $y \in FN$, to find $h: M \rightarrow Z$, $k: N \rightarrow Z$ with $Fh(x) = Fk(y)$. Fix $x \in FM$.

For every object T put $HT = \{t \in FT; \text{there are } h: M \rightarrow Z, k: T \rightarrow Z \text{ with } Fh(x) = Fk(t)\}$; we shall prove that $H = F$. First, H is a subfunctor of F : given $t \in HT$ and given a morphism $p: T \rightarrow T_1$ we have $h: M \rightarrow Z, k: T \rightarrow Z$ with $Fh(x) = Fk(t)$; since p, k have a common domain there exist p', k' with $p'p = k'k$. This proves $Fp(t) \in HT_1$, because $F(k'k)(x) = Fp'(Fp(t))$.



Second, $F - H$ (defined by $(F - H)T = FT - HT$) is a subfunctor of F , as is easily seen. Since F is indecomposable and $F = H \vee (F - H)$, either $F = H$ or $F = F - H$. The latter cannot occur, since $x \in HM$.

(b) Let $\mu, \nu: F \rightarrow F'$ be transformations between non-trivial indecomposable functors. Choose $z \in FM$ arbitrarily and put $x = \mu_M z$, $y = \nu_M z$. Via the previous part of the proof there exist $h, k: M \rightarrow Z$ with $F'h(x) = F'k(y)$. Choose $p: Z \rightarrow T$ with $ph = pk$ and put $t = F(ph)(x)$. Then $\mu_T t = F'(ph)(z) = F'(pk)(z) = \nu_T t$.

(c) is well known.

This concludes the proof.

IV. A corollary

IV.1. Let T be a cocomplete category which has a full subcategory D isomorphic to Set and closed under colimits and finite limits. Then we have

$\text{colim}: T^K \rightarrow T$ is exact $\implies K$ is filtered.

Indeed, if $\text{colim}: T^K \rightarrow T$ is exact so is $\text{colim}: D^K \rightarrow D$, the latter being a restriction of the former one. As $D \sim \text{Set}$, K is filtered by I.2c.

IV.2. The above corollary applies e.g. to the category of

- topological (resp. uniform) spaces,
- graphs,
- unary algebras of a given type

and to T^L for any such T and any small L .

In all of these examples filtered colimits commute with finite limits (as is easily seen) so that we have

$\text{colim}: T^K \rightarrow T$ is exact $\iff K$ is filtered.

R e f e r e n c e s

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