Jiří Adámek; Jan Reiterman
Exactness of the set-valued colim

Commentationes Mathematicae Universitatis Carolinae, Vol. 17 (1976), No. 1, 97--103

Persistent URL: http://dml.cz/dmlcz/105677

Terms of use:
© Charles University in Prague, Faculty of Mathematics and Physics, 1976

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these Terms of use.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project DML-CZ: The Czech Digital Mathematics Library http://project.dml.cz
Abstract: It is well-known that, in the category of sets, filtered colimits commute with finite limits; thus, if $K$ is a filtered small category then the functor $\text{colim} : \text{Set}^K \to \text{Set}$ is exact (i.e. preserves regular epis and finite limits). The converse is proved in the present note and other properties of $\text{colim}$ are investigated and compared with those of $\text{colim} : \text{Ab}^K \to \text{Ab}$ for the category $\text{Ab}$ of Abelian groups.

Key words: Exact colimits, category of sets.

AMS: 08A10, 18B05 Ref. Z.: 2.726

I. Formulation

I.1. The exactness of $\text{colim}$ for $\text{Ab}$ has been investigated by Isbell and Mitchell [2], [3]. In that case $\text{colim}$ is exact iff it preserves equalizers and iff it preserves monics. For the set-valued $\text{colim}$ (i.e. for $\text{colim} : \text{Set}^K \to \text{Set}$) these properties differ. We shall prove namely the following propositions (see part III).

I.2. (a) $\text{colim}$ preserves monics iff every diagram $(x)$
in $K$ is a part of commutative square ($\ast \ast$)

(b) $\text{colim}$ preserves equalizers iff $K$ has filtered components, i.e. iff $K$ fulfils the condition of (a) and for every pair $f, g$ of parallel morphisms there is $k$ with $kf = kg$.

(c) $\text{colim}$ is exact iff $K$ is filtered, i.e. iff $K$ fulfils the conditions of (a), (b) and for every pair $A, B$ of $K$-objects there is $C$ with $\text{Hom}(A, C) \neq \emptyset \neq \text{Hom}(B, C)$.

I.3. This characterization is rather simple in comparison with the Ab case. $\text{Colim}: \text{Ab} \xrightarrow{K} \text{Ab}$ is exact iff the following category $\text{aff } K$ has filtered components: objects of $\text{aff } K$ are just the objects of $K$; morphisms from $A$ to $B$ are those elements $\sum \alpha_i f_i$ of the free Abelian group over $\text{Hom}_K(A, B)$ for which $\sum \alpha_i = 1$, see [3].

I.4. It is easily seen that 1) $\text{aff } K$ has filtered components provided that $K$ has, 2) if $\text{aff } K$ has filtered components then $K$ fulfils the condition of (a). Thus,
denoting \( A = \text{colim} : \text{Ab}^K \rightarrow \text{Ab} \), \( S = \text{colim} : \text{Set}^K \rightarrow \text{Set} \)

we get

\[ S \text{ is exact } \iff S \text{ preserves equalizers } \iff A \text{ is exact } \iff S \text{ preserves monics} \]

None of these implications can be reversed. The counterexamples are easy (according to I.2, I.3) except that to the second implication: for the category \( K \) of finite ordinals and order preserving injections, \( A \) is proved to be exact in [3] but the only component of \( K \) is not filtered.

II. Relation to indecomposable functors

II.1. Colimits in sets are closely related to indecomposability: a functor \( F : K \rightarrow \text{Set} \) is indecomposable if whenever \( F = F_1 \uplus F_2 \) then \( F_1 \) or \( F_2 \) is the constant functor to \( \emptyset \). Notice that \( F \) is indecomposable iff \( \text{colim} \ F \) is a singleton set.

Let us observe that each non-trivial functor \( F : K \rightarrow \text{Set} \) can be decomposed into a sum of its components, i.e. maximal indecomposable subfunctors, \( F = \biguplus_i F_i \). If \( \mu : F \mapsto F' \) is a transformation and \( F' = \biguplus_j F'_j \) is a decomposition of \( F' \) into components then for every \( i \in I \) there is \( c(i) \in J \) with \( \mu(F_i) \subset F'_c(i) \). We have \( \text{colim} \ F = I \), \( \text{colim} \ F' = J \), \( \text{colim} \ \mu = c \). From these observations one can derive the following properties of \( \text{colim} : \text{Set}^K \rightarrow \text{Set} \).

II.2. (a) \( \text{colim} \) preserves monics iff each non-trivial subfunctor of an indecomposable functor \( F : K \rightarrow \text{Set} \) is indecomposable, too.

(b) \( \text{colim} \) preserves equalizers iff indecomposable
functors from $K$ to $\text{Set}$ have always the following "agreement property": for each couple $(\alpha, \beta : F \to F')$ of transformations there is $N$ and $x \in \Phi N$ with $(\alpha_N x = \beta_N x)$.

(c) $\text{colim}$ preserves finite products iff the product of two indecomposable functors from $K$ to $\text{Set}$ is indecomposable, too.

II.3. The exactness of $\text{colim}$ in the $\text{Ab}$ case can be also characterized analogously [1]: $\text{colim}: \text{Ab}^K \to \text{Ab}$ is exact iff the agreement property from (b) holds for all couples of endo-transformations of indecomposable functors from $K$ to $\text{Set}$; equivalently, iff each endo-transformation $\alpha : F \to F$ of an indecomposable functor $F : K \to \text{Set}$ has a fixed point (i.e. $x$ in some $\Phi N$ with $\alpha_N x = x$).

III. Proof

III.1. Necessities in I.2 follow from II.2 if we take into account that

(a) the subfunctor $F$ of $\text{Hom}(M,-)$ generated by $f : M \to C$, $g : M \to D$ must be indecomposable (then we have $f' : C \to E$, $g' : D \to E$ with $f' f = g' g$),

(b) the transformations $\text{Hom}(f,-)$, $\text{Hom}(g,-)$: $\text{Hom}(N,-) \to \text{Hom}(M,-)$ must coincide at some $k \in \text{Hom}(N,C)$; and all monics are equalizers in $\text{Set}^K$,

(c) the product $\text{Hom}(M,-) \times \text{Hom}(N,-)$ must be non-trivial.

III.2. Sufficiencies. (a) Let $F : K \to \text{Set}$ be an indecomposable functor. To prove that all subfunctors of $F$
are indecomposable it suffices, for given \( x \in FM \), \( y \in FN \), to find \( h: M \to Z \), \( k: N \to Z \) with \( Fh(x) = Fk(y) \). Fix \( x \in FM \).

For every object \( T \) put \( HT = \{ t \in FT \} \); there are \( h: M \to Z \), \( k: T \to Z \) with \( Fh(x) = Fk(t) \); we shall prove that \( H = F \). First, \( H \) is a subfunctor of \( F \): given \( t \in HT \) and given a morphism \( p: T \to T_1 \) we have \( h: M \to Z \), \( k: T \to Z \) with \( Fh(x) = Fk(t) \); since \( p, k \) have a common domain there exist \( p', k' \) with \( p'p = k'k \). This proves \( Fp(t) \in HT_1 \), because \( F(k'k)(x) = Fp'(Fp(t)) \).

Second, \( F - H \) (defined by \( (F - H)T = FT - HT \)) is a subfunctor of \( F \), as is easily seen. Since \( F \) is indecomposable and \( F = H \lor (F - H) \), either \( F = H \) or \( F = F - H \). The latter cannot occur, since \( x \in FM \).

(b) Let \( (\alpha, \beta): F \to F' \) be transformations between non-trivial indecomposable functors. Choose \( x \in FM \) arbitrarily and put \( x = (\alpha y)z \), \( y = \beta y' \). Via the previous part of the proof there exist \( h, k: M \to Z \) with \( F'h(x) = F'k(y) \).

Choose \( p: Z \to T \) with \( ph = pk \) and put \( t = p(ph)(x) \). Then \( \alpha_t = F'(ph)(z) = \beta_t(z) = \beta_t \).

(c) is well known.
This concludes the proof.

IV. A corollary

IV.1. Let $T$ be a cocomplete category which has a full subcategory $D$ isomorphic to $\text{Set}$ and closed under colimits and finite limits. Then we have

$$\text{colim}: T^K \to T \text{ is exact} \implies K \text{ is filtered.}$$

Indeed, if $\text{colim}: T^K \to T \text{ is exact}$ so is $\text{colim}: D^K \to D$, the latter being a restriction of the former one. As $D \simeq \text{Set}$, $K$ is filtered by 1.2c.

IV.2. The above corollary applies e.g. to the category of
- topological (resp. uniform) spaces,
- graphs,
- unary algebras of a given type
and to $T^L$ for any such $T$ and any small $L$.

In all of these examples filtered colimits commute with finite limits (as is easily seen) so that we have

$$\text{colim}: T^K \to T \text{ is exact} \iff K \text{ is filtered.}$$

References


