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Recognizable filters and ideals

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Abstract: Necessary and sufficient conditions are obtained for filters, ultrafilters, and ideals over a free monoid to be recognizable by finite branching automata.

Key-words: Filter, ultrafilter, ideal, formal language, recognizable family of languages, finite branching automaton.

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Recognizable families of formal languages were introduced and studied in connection with formalization of certain aspects of state-space problem solving by means of finite branching automata (see [1]). In that formalism languages (sets of strings over a finite alphabet $\Sigma$) represent plans of behaviour incorporating branching. In an earlier paper [2] we obtained a series of results concerning recognizable families of languages as well as their interesting subclass, the well-recognizable families (recognizable families with recognizable complements).

In the present paper we focus on a particular problem concerning the relationship between recognizable families of languages on one hand and filters and ideals over the free monoid $\Sigma^*$ on the other hand. The concept of a filter, and
its dual notion of an ideal, are important in various areas of mathematics: filters over $\Sigma^*$ were discussed in [3] especially in connection with concatenation of families.

Here we shall obtain necessary and sufficient conditions for filters and ideals over $\Sigma^*$ to be recognizable. We shall also show that a recognizable filter is well-recognizable iff it is an ultrafilter. Thus concepts approached from completely different directions appear surprisingly interrelated.

In the present context an alphabet $\Sigma$ is an arbitrary finite non-empty set of objects called letters (usually denoted $a, b, c, \ldots$). We denote by $\Sigma^*$ the set of all finite sequences of letters (the free monoid generated by $\Sigma^*$ under concatenation). The elements of $\Sigma^*$ are called strings and usually denoted $u, v, w, \ldots$. The unit element in $\Sigma^*$ is the empty string $\lambda \in \Sigma^*$. We denote $\Sigma^*_\lambda = \Sigma^* \cup \{\lambda\}$. For $u \in \Sigma^*$, $\lg(u)$ denotes the length of $u$ (the number of occurrences of letters in $u$). In particular, $\lg(\lambda) = 0$. For $u, v \in \Sigma^*$, $u \equiv v \equiv (\exists w \in \Sigma^*) (uw = v)$. $P(\Sigma^*)$ is the set of all subsets of $\Sigma^*$, $L(\Sigma)$ is the set of all non-empty subsets of $\Sigma^*$, elements of $L(\Sigma)$ are called languages (usually denoted $L$).

Any $X \subseteq L(\Sigma)$ will be called a family of languages (over $\Sigma$). Note that we admit empty family of languages but not families with empty element. We shall use the usual set-theoretical operations, union ($\cup$), intersection ($\cap$) and complement ($\overline{X} = \{L \in L(\Sigma) : L \not\in X\}$). For $u \in \Sigma^*$ and $L \in L(\Sigma)$ we define:

1) the derivative of $L$ with respect to $u$
Let us note that, as it is known from classical automata theory, a language \( L \) is regular (i.e., recognizable by a classical finite automaton) iff the set \( \{ \partial_u L; u \in \Sigma^* \} \) is finite. The reader unfamiliar with the automata theory may
consider this fact as a definition of a regular language. 
(Note that in the classical automata theory $\emptyset$ is also a regular language.)

For the definition and basic properties of filters, see e.g. [4] IV, 8, p. 193-196.

**Definition 4.** A filter $F$ over $\Sigma^*$ is a non-empty subset of $\mathcal{P}(\Sigma^*)$ satisfying:

1) $\emptyset \notin F$;
2) if $A, B \in F$ then $A \cap B \in F$;
3) if $A \in F$ and $A \subseteq B$ then $B \in F$.

In this paper we assume $\Sigma$ to be a fixed alphabet and shall call filters over $\Sigma^*$ simply filters.

Since $\emptyset \notin F$ every filter is a subset of $\mathcal{P}(\Sigma)$ and we can look at it as a family of languages. For any $L \in \mathcal{L}(\Sigma)$ the family $\{L'; L \subseteq L';\}$ is clearly a filter over $\Sigma^*$. Over an infinite set there exist also filters of other types (here e.g. family of all languages with finite complements).

**Definition 5.** A filter of the type $\{L'; L \subseteq L';\}$ is called principal and will be written $P_L$.

It is easy to show that a filter $F$ is principal iff $\cap F \in F$.

**Definition 6.** A filter $F$ is called an ultrafilter if $F$ is a maximal filter, i.e. there exists no filter $F'$ such that $F \subseteq F'$.

Again it is easy to show that a principal filter over $\Sigma^*$ is an ultrafilter iff it is of the form $P_{\{u}\}$ for some $u \in \Sigma^*$.

**Definition 7.** A filter $X$ is a recognizable (well-recog-
nizable) filter if the family $X$ is recognizable (well-recog-
nizable). Analogically we define a recognizable, resp.
well-recognizable ultrafilter.

**Theorem 8.** A filter over $\Sigma^*$ is recognizable iff it
is a principal filter of the form $F_L$ where $L$ is a regular lan-
guage.

**Proof.** First we show that every principal filter is
self-compatible.
Let $L \in C(F_L)$, for the sake of contradiction we shall assu-
me that $L \notin F_L$, i.e. there exists $u \in L$ such that $u \notin L'$. By
the definition of $C$-closure there must exist $L_u \in F_L$ such
that particularly $\lambda \in Fst_{\lambda} (\partial_u L') \Rightarrow \lambda \in Fst_{\lambda} (\partial_u L_u)$ and
thus $u \in L' \equiv u \in L_u$. But $u \in L_u$ because $L \subseteq L_u$ and thus also
$u \in L'$, which contradicts the assumption.
Furthermore, for any $u \in \Sigma^*$,

$$\partial_u F_L = \partial_u \{ L' : L \subseteq L' \} = \{ L_u : \partial_u L \subseteq L_u \} .$$

Thus $\partial_u F_L = \partial_u F_L = \partial_u L = \partial_u L'$, i.e., $F_L$ is a finitely
derivable family iff $L$ is a regular language.

Now we have known that a principal filter $F_L$ is recognizab-
le iff $L$ is a regular language. It remains to show that eve-
ry recognizable filter $F$ must be principal, i.e. that $\bigcap F \in 
\in F$. Let $F$ be a recognizable filter. First we show that if
$\bigcap F \subseteq L$ and $L$ is a complete language then $L \in F$ (for the de-
inition of a complete language see e.g. [5], p. 47). In our
notation $L$ is complete language iff $( \forall u \in \Sigma^*)( \Sigma \subseteq 
\subseteq Fst_{\lambda} (\partial_u L))$. For $u \in L$,

$$Fst_{\lambda} (\partial_u L) = \Sigma_{\lambda} = Fst_{\lambda} (\partial_u \Sigma^*)$$
and for \( u \notin L \),

\[ Fst_\Lambda ( \partial_u L ) = \Sigma^* = Fst_\Lambda ( \partial_u ( \Sigma^* - \{ u \} ) ) . \]

But necessarily \( \Sigma^* \in F(F \text{ is non-empty}) \) and if \( u \notin L \) then by the assumption \( u \notin L \cap F \), i.e. there exists \( L' \in F \) such that \( u \notin L' \) and since \( L' \in \Sigma^* - \{ u \} \) then by the property 3) of filter also \( \Sigma^* - \{ u \} \in F \). Therefore \( L \in C(F) \) and thus \( L \in F \) by the assumption about recognizability of \( F \). Now it is easy to choose arbitrary two complete languages \( L_1 \) and \( L_2 \) for which \( L_1 \cap L_2 = \bigcap F \).

We have shown that \( L_1 \in F \) and \( L_2 \in F \) and thus also \( L_1 \cap L_2 = \bigcap F \in F \) (property 2)).

**Theorem 9.** A principal filter of the form \( F_L \) is well-recognizable iff it is an ultrafilter.

**Proof.** We have stated (cf. [4], p. 196) that principal filter is an ultrafilter iff it is of the form \( F_{\{ u \} } \) for \( u \in \Sigma^* \). By the preceding theorem \( F_{\{ u \} } \) is recognizable.

Clearly for every \( v \in \Sigma^* \) such that \( lg(v) > lg(u) \), \( \partial_v F_{\{ u \} } = \bigcap (\Sigma^*) \). Thus \( F_{\{ u \} } \) is finitely derivable and furthermore \( C(F_{\{ u \} }) = \overline{F_{\{ u \} } } \) because for every \( L \in F_{\{ u \} } \), \( \Lambda \in Fst_\Lambda ( \partial_u L ) \), while for any \( L \in F_{\{ u \} } \), \( \Lambda \notin Fst_\Lambda ( \partial_u L ) \). Thus also \( F_{\{ u \} } \) is recognizable and so \( F_{\{ u \} } \) is a well-recognizable family.

Now let us assume, for contradiction, that \( F_L \) is not an ultrafilter, i.e. there exists \( v, w \in L \) such that \( w \neq v \). Thus by the definition of \( F_L \) we have \( \Sigma^* - \{ v \} \in \overline{F_L} \) and \( \Sigma^* - \{ w \} \in \overline{F_L} \). But for any \( u \in \Sigma^* \) we have \( u \neq v \) \( \implies \)

\[ \implies Fst_\Lambda ( \partial_u ( \Sigma^* - \{ v \} ) ) = \Sigma^*_\Lambda = Fst_\Lambda ( \partial_u ( \Sigma^* - \{ w \} ) ) ; \]
\[ u = v \implies Fst_\Lambda ( \partial_u ( \Sigma^* ) ) = \Sigma^*_\Lambda = Fst_\Lambda ( \partial_u ( \Sigma^* - \{ w \} ) ) . \]
Thus $\Sigma^* \in C(F_L)$ and since $\Sigma^* \notin \overline{F_L}$ we have $C(F_L) \neq F_L$ and so $F_L$ is not a well-recognizable filter.

Q.e.d.

In the paper [2] we have shown that to every nontrivial well-recognizable family $X$ there exists exactly one string $u_X \in \Sigma^*$ such that the families $\partial^X$ are trivial (i.e. $\emptyset$ or $\mathcal{L}(\Sigma)$) for all $v \notin u_X$ while they are nontrivial and mutually distinct for all $v \leq u_X$. We have called $u_X$ the characteristic string of a family $X$ because it uniquely determines $X$ regarding the algebraic decomposition of $X$ to a finite number of basic families and regarding the (minimal) number of states of a branching automaton recognizing $X$. It can be easily seen that for an ultrafilter $F_{u^3}$, the string $u$ satisfies the above conditions and thus $u_{F_{u^3}} = u$ (i.e. there exists finite branching automaton with $(\lg(u) + 2)$ states recognizing the family $F_{u^3}$ - cf. [2]).

The preceding theorems showed us an interesting relationship between recognizable families and filters, as well as between well-recognizable families and ultrafilters.

We shall now turn to a dual notion to that of a filter, namely the ideal. We obtain results analogical to those concerning filters. Our definition of an ideal is a slight modification of that from [6], p. 132.

**Definition 10.** A non-empty set $I$ of subsets of $\Sigma^*$ is an ideal over $\Sigma^*$ if

1) $\Sigma^* \notin I$;

2) if $A, B \in F$ then $A \cup B \in I$;

3) if $A \in I$ and $B \subseteq A$ then $B \in I$.

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Again we shall call ideals over $S^*$ simply ideals.

We want to talk about recognizable ideals. However, since always $\emptyset \in I$ no ideal is a "family" in our sense. We shall therefore use the following definition.

**Definition 11.** We say that an ideal $I$ is a recognizable ideal if $I - \{ \emptyset \}$ is a recognizable family of languages.

Similarly as in the case of principal filters we have again principal ideals of the form $I_A = \{ B; B \leq A \}$, where $A \subseteq S^*$.

An ideal is principal iff $\bigcup I \in I$.

**Theorem 12.** An ideal $I$ is recognizable iff it is a principal ideal of the form $I_A$, where $A$ is a regular language (possibly empty), $A \neq S^*$.

**Proof.** If $A = \emptyset$, $I_A = \{ \emptyset \} = \emptyset$ is a trivial recognizable family. If $A = L \in \mathcal{L}(S^*)$, then in the same way as in Theorem 8 one can show that $I_L = \{ \emptyset \}$ is self-compatible, as well as that it is finitely derivable iff $L$ is finitely derivable.

It suffices to show that a recognizable ideal is principal, i.e. that $\bigcup I \in I$.

If $\bigcup I = \emptyset$ then $I = I_\emptyset$ is principal.

Otherwise we put $\bigcup I = L$ and show that $L$ is in the closure of $I - \{ \emptyset \}$. Since for every $L' \in I$, $L' \subseteq L$ and since an ideal is closed under finite union, for every $u \in S^*$ there surely exists $L_u \in I$ satisfying the conditions:

a) $(\forall v \in S^*) \left[ \lg(v) = \lg(u) + 1 \Rightarrow (v \in \text{Pref}(L) \iff v \in \text{Pref}(L_u)) \right]$;
b) $u \in L \iff u \in L_u$.

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However, then $Fst_A(\emptyset_L) = Fst_A(\emptyset_L')$. Thus $L \in C(I - \emptyset) = I - \emptyset$, i.e. $I$ is a principal ideal.

Q.e.d.

References