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FINITENESS CONDITIONS ON EDZ - VARIETIES

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Abstract: We shall study conditions for a given EDZ-variety to be locally finite and to be generated by a finite algebra. These two properties are algorithmically decidable. An EDZ-variety of a finite type is generated by a finite algebra iff it is locally finite and finitely axiomatized.

Key words: Variety, locally finite, generated.

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The study of EDZ-varieties (varieties of universal algebras with equationally definable zeros) provides us with various counterexamples, suitable in many respects. Moreover, EDZ-varieties are worth themselves of a special attention. Their investigation was begun in [1] and [2]. In the present paper we shall be concerned with the finiteness and generability by a finite algebra. We shall preserve the terminology of [1] (with a slight modification regarding the length of a term). Some terminology and notations will be listed now.

The set of variables is denoted by $X = \{x_1, x_2, \dots\}$. If Δ is a type (i.e. a set of operation symbols), we denote by W_Δ the algebra of Δ-terms. For every $t \in W_\Delta$ let

$\lambda(t), \lambda'(t)$ denote the numbers defined as follows: if t is a variable or a constant, then $\lambda(t) = \lambda'(t) = 1$; for $t = F(t_1, \ldots, t_{n_F})$ put $\lambda(t) = 1 + \lambda(t_1) + \ldots + \lambda(t_{n_F})$ and $\lambda'(t) = \lambda'(t_1) + \ldots + \lambda'(t_{n_F})$. In this paper $\lambda'(t)$ is called the length of t.

The definition of an irreducible set of Δ-terms, of an EDZ-variety and related concepts, as well as their basic properties, are contained in [1] and repeated in [2].

A variety K of universal algebras is called locally finite if every finitely generated algebra from K is finite. It is well-known (see e.g.[3]) that if a variety is generated by a finite algebra, then it is locally finite. The converse is not true (a counterexample could be easily derived from results of this paper).

Let J be an arbitrary non-empty set of Δ-terms. For every positive integer n we define a Δ-algebra W_n^J as follows: its underlying set is the set $W_n - \Phi(J) \cup \{0\}$, where W_n is the subalgebra of W generated by $\{x_1, \ldots, x_n\}$; if $F \in \Delta$, $t_1, \ldots, t_{n_F} \in W_n - \Phi(J)$ and $F(t_1, \ldots, t_{n_F}) \notin \Phi(J)$, then we put $F_{W_n}(t_1, \ldots, t_{n_F}) = F(t_1, \ldots, t_{n_F})$; in other cases we put $F_{W_n}(t_1, \ldots, t_{n_F}) = 0$. It is easy to see that W_n^J is the Z_J-free algebra over $\{x_1, \ldots, x_n\}$.

Let us define a set $\overline{W_\Delta}$ by $t \in \overline{W_\Delta}$ iff t contains no constants and whenever $F(u_1, \ldots, u_{n_F})$ is a subterm of t, then at most one of the terms $u_1, \ldots, u_{n_F}$ is not a variable; now for every $t \in \overline{W_\Delta}$ we define a finite sequence $\sigma(t)$ as follows: if t is a variable, then put $\sigma(t) = \langle t \rangle$; if $t = F(y_1, \ldots, y_{n_F})$, where $y_1, \ldots, y_{n_F}$ are variables, then put

$\sigma(t) = \langle y_1,\dots,y_{n_F}\rangle$; if $t = F(y_1,\dots,y_{j-1},u,y_{j+1},\dots$ $\dots,y_{n_F})$, where u is not a variable and $\sigma(u) = \langle z_1,\dots$ $\dots,z_m\rangle$, put $\sigma(t) = \langle z_1,\dots,z_m,y_1,\dots,y_{n_F}\rangle$. It is obvious that if $\sigma(t) = \langle y_1,\dots,y_n\rangle$, then $n = \lambda'(t)$.

For every $J \subseteq W_\Delta$ we define two subsets J' and J'' of J as follows: $t \in J'$ if $t \in J$, t contains no constants and no variable has more than one occurence in t; $J'' = J' \cap \overline{W_\Delta}$.

For every Δ-term t let o(t) denote the positive integer defined in this way: if t is a variable or a constant, then $o(t) = 1$; if $t = F(t_1,\dots,t_{n_F})$, then $o(t) = \max\{o(t_1),\dots,o(t_{n_F})\} + 1$.

Proposition 1. Let J be an irreducible set of Δ-terms. The variety Z_J is locally finite iff W_1^J is finite and for every positive integer n there exists a positive integer k_n such that $\{t \in W_n;\ \lambda'(t) \geq k_n\} \subseteq \Phi(J)$ and $\{F \in \Delta;\ n_F = n,\ F(x_1,\dots,x_{n_F}) \notin \Phi(J)\}$ is finite.

Proof is easy.

Proposition 2. Let J be an irreducible set of Δ-terms. The variety Z_J is generated by a finite algebra iff it is locally finite and there exists a positive integer m such that $\{t \in W_\Delta;\ \lambda'(t) \geq m\} \subseteq \Phi(J)$.

Proof. Let Z_J be generated by a finite algebra. It is easy to see that Z_J is locally finite and that Z_J is generated by W_n^J for some positive integer n. Since W_n^J is finite, there exists a positive integer m such that $\{t \in W_n;\ \lambda'(t) \geq m\} \subseteq$ $\subseteq \Phi(J)$.

Let t be an arbitrary Δ-term of length $\geq m$; it is enough to prove $t \in \Phi(J)$. There exists a term of length $\geq m$ such that $t \neq v$. If φ is an arbitrary homomorphism of W_Δ into W_n^J, then evidently $\varphi(t) = \varphi(v) = 0$. Hence the identity $\langle t,v \rangle$ is satisfied in W_n^J; since W_n^J generate J, it is satisfied in Z_J and thus $t \in \Phi(J)$.

Conversely, let Z_J be locally finite and every term of length $\geq m$ belong to $\Phi(J)$. The algebra W_{m-1}^J is finite and it is enough to show that Z_J is generated by W_{m-1}^J. This will be proved if we derive a contradiction from the following assumption: there exist Δ-terms u, v such that $u \neq v$, the identity $\langle u,v \rangle$ is satisfied in W_{m-1}^J and $u \notin \Phi(J)$.

Denote by $y_1,\dots,y_k$ the variables contained in u. Since $u \notin \Phi(J)$, we have $k < m$. There exists an automorphism α of W_Δ such that $\{\alpha(y_1),\dots,\alpha(y_l)\} \subseteq \{x_k,\dots,x_{m-1}\}$, so that $\alpha(u) \in W_{m-1}^J$. Evidently $\alpha(u) \neq \alpha(v)$ and the identity $\langle \alpha(u), \alpha(v) \rangle$ is satisfied in W_{m-1}^J. Let φ be the homomorphism of W_Δ onto W_{m-1}^J defined as follows: $\varphi(x_1) = x_1,\dots$ $\dots, \varphi(x_{m-1}) = x_{m-1}, \varphi(x_m) = \varphi(x_{m+1}) = \dots = 0$. Evidently $\varphi(t) = t$ for all $t \in W_{m-1}^J - \{0\}$ and $\varphi(t) = 0$ for all other t.

Since $\langle \alpha(u), \alpha(v) \rangle$ is satisfied in W_{m-1}^J, $\varphi(\alpha(u)) = \varphi(\alpha(v))$, i.e. $\alpha(u) = \varphi(\alpha(v))$. This implies $\varphi(\alpha(v)) \neq 0$ and thus $\varphi(\alpha(v)) = \alpha(v)$. We get $\alpha(u) = \alpha(v)$ and consequently $u = v$, a contradiction.

<u>Proposition 3</u>. Let J be an irreducible set of Δ-terms. Then for every integer $n \geq 1$ the following conditions are equivalent:

i) Z_J is locally finite and $\{t \in W_\Delta ; \lambda'(t) \geq n\} \subseteq \Phi(J)$;

ii) $Z_{J'}$ is locally finite and $\{t \in W_\Delta ; \lambda'(t) \geq n\} \subseteq \Phi(J')$;

iii) the algebra $W_1^{J'}$ is finite and $\lambda'(t) < n$ for all terms $t \in W_1^{J'} - \{0\}$.

Proof. i) $\Longrightarrow$ ii). Let $t \in W_\Delta$ and $\lambda'(t) \geq n$. Evidently there exists a term $s \in W'_\Delta$ such that $s \leq t$ and $\lambda'(s) = \lambda'(t)$; since $\lambda'(s) \geq n$, we have $s \in \Phi(J)$ by i) and so $\varphi(w)$ is a subterm of s for some $w \in J$ and some endomorphism φ of W_Δ. Clearly $w \in J'$ and thus $t \in \Phi(J')$. We have proved $\{t \in W_\Delta ; \lambda'(t) \geq n\} \subseteq \Phi(J')$. The rest is easy by Proposition 1.

ii) $\Longrightarrow$ iii) is obvious.

iii) $\Longrightarrow$ i). Let φ be the endomorphism of W_Δ defined by $\varphi(x_i) = x_1$ for all $i = 1,2,\ldots$.

Let $t \in W_\Delta$ and $\lambda'(t) \geq n$. We have $\varphi(t) \in W_1$ and $\lambda'(\varphi(t)) = \lambda'(t)$. There exist an endomorphism ψ of W_Δ and a term $u \in J'$ such that $\psi(u)$ is a subterm of $\varphi(t)$. Put var $u = \{y_1,\ldots,y_{\lambda'(u)}\}$. From the definition of φ it is easy to see that there exist subterms $t_1,\ldots,t_{\lambda'(u)}$ of t such that $\varphi(t_i) = \psi(y_i)$ and such that $\psi'(u)$ is a subterm of t, if ψ' is an endomorphism of W_Δ such that $\psi'(y_i) = t_i$. Hence $t \in \Phi(J') \subseteq \Phi(J)$.

Similarly if $F(x_1,\ldots,x_{n_F}) \notin \Phi(J)$, then $\varphi(F(x_1,\ldots,x_{n_F})) \in W_1^{J'}$. The local finiteness of Z_J follows now from Proposition 1.

Corollary. Let J be an irreducible set of Δ-terms and let the variety Z_J be locally finite. Then Z_J is generated by a finite algebra iff $Z_{J'}$ is locally finite.

Proof. Follows from Propositions 2 and 3.

Proposition 4. Let J be a finite irreducible set of

Δ-terms. Suppose that the variety Z_J is non-trivial and locally finite. Then Δ is finite and Z_J is generated by a finite algebra.

<u>Proof.</u> If Δ were infinite, then there would exist a symbol $F \in \Delta$ ($n_F \neq 0$) such that no term from J contains a subterm of the form $F(u_1,\dots,u_{n_F})$. Consequently e.g. the algebra $W^J_{n_F}$ would contain infinitely many terms $t_1,t_2,t_3,\dots$, where $t_1 = F(x_1,\dots,x_{n_F}),\dots,t_{n+1} = F(t_n,\dots,t_n)$, a contradiction.

Put $k = 2 + \max\{n_F;\ F \in \Delta\}$ and for every positive integer n put $S_n = \{t \in W''_\Delta;\ o(t) = n\}$.

Suppose first that for every positive integer n there exists a term $t_n \in S_n - \Phi(J'')$. Put $T = \{t_1,t_2,\dots\}$ and $s = \max\{\lambda'(t);\ t \in J\}$. Since Z_J is locally finite, there exists an r such that $\{t \in W_s;\ \lambda'(t) \geq r\} \subseteq \Phi(J)$.

Let us define a set T_s of Δ-terms by $t \in T_s$ iff the following two conditions are satisfied:

a) $t \in W_s \cap \overline{W_\Delta}$,

b) if $\sigma(t) = \langle y_1,\dots,y_p\rangle$ and $y_i = y_j$ for $i,j \in \{1,\dots,p\}$, then $i \equiv j \pmod{s}$.

Let us prove that if $t \in T_s$ and $\lambda'(t) \geq r$, then $t \in \Phi(J'')$. We have evidently $t \in \Phi(J)$, so that there exist a term $u \in J$ and an endomorphism ψ of W_Δ such that $\psi(u)$ is a subterm of t. It is not difficult to prove (using $t \in T_s$) that $u \in J'$. Now $u \in J''$ is easy and so $t \in \Phi(J'')$.

There exist a number $n \geq r$ and a term $t \in T_s$ such that $\sigma(t) = \langle x_1,\dots,x_n\rangle$ for some automorphism α of W_Δ. Let us define an endomorphism φ of W_Δ in this way:

$\varphi(x_i) = x_j$, where $j \in \{1,\dots,s\}$ and $i \equiv j (\mathrm{mod}\ s)$. Evidently $\varphi(\alpha(t)) \in T_s$ and $\lambda'(\varphi(\alpha(t))) = n$, so that $\varphi(\alpha(t)) \in \Phi(J'')$. Similarly as in the proof of Proposition 3 (iii) $\Rightarrow$ i)) it can be proved that $\alpha(t) \in \Phi(J'')$ and consequently $t \in \Phi(J'')$, a contradiction with the assumption $t \notin \Phi(J'')$. Denote by n the smallest number such that $S_n \subseteq \Phi(J'')$. By Proposition 2 it is enough to show that if $t \in W_\Delta$ and $\lambda'(t) \geq k^{n-1}$, then $t \in \Phi(J'') \subseteq \Phi(J)$.

Evidently $n \geq 2$, since Z_J is non-trivial; we shall define sets $P_1,\dots,P_{n-1}$ as follows:
we have $t = F_1(u_1^1,\dots,u_{n_{F_1}}^1)$. If $n = 2$, put $P_1 = \{u_1^1,\dots,u_{n_{F_1}}^1\}$. If $n \geq 3$, then there exists a number $j_1 \in \{1,\dots,n_{F_1}\}$ such that $\lambda'(u_{j_1}^1) \geq k^{n-2}$; put $P_1 = \{u_1^1,\dots,u_{j_1-1}^1, u_{j_1+1}^1,\dots,u_{n_{F_1}}^1\}$. Again we have $u_{j_1}^1 = F_2(u_1^2,\dots,u_{n_{F_2}}^2)$. If $n = 3$, put $P_2 = P_1 \cup \{u_1^2,\dots,u_{n_{F_2}}^2\}$. If $n \geq 4$, then there exists a number $j_2 \in \{1,\dots,n_{F_2}\}$ such that $\lambda'(u_{j_2}^2) \geq k^{n-3}$; put $P_2 = P_1 \cup \{u_1^2,\dots,u_{j_2-1}^2, u_{j_2+1}^2,\dots,u_{n_{F_2}}^2\}$. If we have defined $P_1, P_2,\dots,P_{n-2}$, put $P_{n-1} = P_{n-2} \cup \{u_1^{n-1},\dots,u_{n_{F_{n-1}}}^{n-1}\}$ and let us define terms $t^{(n-1)},\dots,t^{(1)}$ in this way:

$t^{(n-1)} = F_{n-1}(x_1,\dots,x_{n_{F_{n-1}}})$, $t^{(n-2)} = F_{n-2}(y_1,\dots,y_{j_{n-2}-1}, t^{(n-1)}, y_{j_{n-2}},\dots,y_{n_{F_{n-2}}-1})$, where $y_1,\dots,y_{n_{F_{n-2}}-1}$ are pairwise different variables not occuring in $t^{(n-1)}$,

$t^{(1)} = F_1(z_1,\dots,z_{j_1-1}, t^{(2)}, z_{j_1},\dots,z_{n_{F_1}-1})$, where $z_1,\dots,z_{n_{F_1}-1}$ are pairwise different variables not occuring in

$t^{(2)}$. Evidently $t^{(1)} \in S_n$ and $t \in \Phi(\{t^{(1)}\}) \subseteq \Phi(J'')$.

Proposition 5. Let J be an irreducible set of terms of a finite type Δ and let Z_J be generated by a finite algebra. Then J is finite.

Proof. Put $k = \max\{n_F;\ F \in \Delta\}$ and let n be the smallest positive integer such that $\{t \in W_\Delta;\ \lambda'(t) \geq n\} \subseteq \Phi(J)$. Let us denote by T the set of Δ-terms $t \in W_{n+k} \cap \Phi(J)$ such that $\lambda'(t) \leq n + k$. Obviously T is finite, so that there exists a finite irreducible subset $S \subseteq T$ such that $\Phi(S) = \Phi(T)$.

Let us prove by induction on $\lambda(t)$ that $t \in \Phi(J)$ implies $t \in \Phi(T)$. If $t \in \Phi(J)$ and $\lambda'(t) \leq n + k$, then there is an automorphism α of W_Δ with $\alpha(t) \in W_{n+k}$; we have $\alpha(t) \in W_{n+k} \cap \Phi(J)$, i.e. $\alpha(t) \in T$, so that $t \in \Phi(T)$.

Let $\lambda'(t) > n + k$ and $t \in \Phi(J)$. There exist a symbol G and terms $y_1, \ldots, y_{n_G}$ such that $G(y_1, \ldots, y_{n_G})$ is a subterm of t and every y_i is either a variable or a constant. Let z be a variable not contained in t. If we replace precisely one occurence of $G(y_1, \ldots, y_{n_G})$ in t by z, we obtain a new term s. Evidently $\lambda(s) < \lambda(t)$ and $\lambda'(s) \geq \lambda'(t) - k + 1 > n$, so that $s \in \Phi(J)$. By the induction assumption $s \in \Phi(T)$. However $s \leq t$, so that $t \in \Phi(T)$, too.

We have proved $\Phi(J) \subseteq \Phi(T)$. Since $\Phi(T) \subseteq \Phi(J)$ is obvious, we get $\Phi(J) = \Phi(T) = \Phi(S)$. Since every two irreducible generating subsets of $\Phi(J)$ have the same cardinality, J has the same cardinality as S and consequently J is finite.

Theorem 1. Let J be an irreducible set of terms of a finite type Δ . Then the variety Z_J is generated by a finite algebra iff Z_J is locally finite and J is finite.

Proof. Follows from Propositions 4 and 5.

For every positive integer p and for every $J \subseteq W_\Delta$ we define $S_p = \{t \in W_\Delta ; o(t) = p\}$,

$U_p = \{t \in W_\Delta ; o(t) = p, \quad \sigma(t) = \langle x_1, \ldots, x_{\lambda'(t)} \rangle$,

$J_p = U_p \cap \Phi(J'')$.

Proposition 6. Let J be a finite irreducible set of terms of a finite type Δ and let the variety Z_J be locally finite. If $k = \max\{n_F; F \in \Delta\} + 2$, $p = \max\{o(t); t \in J''\}$, $r = \text{card } U_p$, $q = \text{card } J_p$, then $\{t \in W_\Delta ; \lambda'(t) \geq k^{p+r-(q+1)}\} \subseteq \Phi(J'')$.

Proof. For every $t \in S_p$ we shall construct a term $u \in \Phi(J'')$ as follows.

If $t \in \Phi(J'')$, put $u = t$. If $t \notin \Phi(J'')$, then for an arbitrary symbol $G \in \Delta$ such that $n_G \neq 0$ we define $t_1 = G(u_1, \ldots, u_{n_G})$, where $\{u_1, \ldots, u_{n_G}\} = \{y_1, \ldots, y_{n_G-1}, t\}$ and $y_1, \ldots, y_{n_G-1}$ are arbitrary variables.

There exist a symbol $F \in \Delta$ and variables $z_1, \ldots, z_{n_F}$ such that $F(z_1, \ldots, z_{n_F})$ is a subterm of t_1. Let us replace this subterm by x_1 and all other occurences of variables in t_1 which are not contained in this subterm by $x_2, x_3, \ldots$, so that the new term t_1' is such that $\sigma(t_1') = \langle x_1, \ldots, x_{\lambda'(t_1')} \rangle$. Obviously $t_1' \in U_p$; since $t \notin \Phi(J'')$, we have $t_1 \in \Phi(J'')$ iff $t_1' \in J_p$.

If $t_1 \in \Phi(J'')$, put $u = t_1$. If $t_1 \notin \Phi(J'')$, then for an arbitrary symbol $H \in \Delta$ such that $n_H \neq 0$ we define $t_2 = H(v_1,\dots,v_{n_H})$, where $\{v_1,\dots,v_{n_H}\} = \{w_1,\dots,w_{n_H-1},t_1\}$ and $w_1,\dots,w_{n_H-1}$ are arbitrary variables.

There exists a symbol $E \in \Delta$ such that $E(\dots,F(z_1,\dots,z_{n_F}),\dots)$ is a subterm of t_2. Let us replace this subterm by x_1 and all other occurences of variables in t_2 which are not contained in this subterm by $x_2,x_3,\dots$, so that the new term t_2' is such that $\sigma(t_2') = \langle x_1,\dots,x_{\lambda'(t_2')}\rangle$.

Again $t_2' \in U_p$ and $t_2 \in \Phi(J'')$ iff $t_2' \in J_p$. If $t_2 \in \Phi(J'')$, put $u = t_2$. If $t_2 \notin \Phi(J'')$, we can define analogously terms $t_3, t_3', \dots$.

Put $V = \{t_1,t_2,\dots\}$. We shall show that $t_i' \neq t_j'$, if $i \neq j$. In the contrary case let $\langle i,j\rangle$ be pair the first such that $i < j$ and $t_i' = t_j'$. We can define terms $u_{j+1},u_{j+2},\dots$ such that for every positive integer m $o(u_{j+m}) = p + j + m$ and $u_{j+m}' = t_n'$, where $i \leq n < j$ iff $m \equiv n (\mathrm{mod}\ j - i)$. If $t_{i+1} = F(y_1,\dots,t_i,\dots,y_{n_F-1})$, then we put $u_{j+1} = F(y_1,\dots,t_j,\dots,y_{n_F-1})$ and if u_{j+m} is already defined, $m \equiv n (\mathrm{mod}\ j - i)$ for some n $(i \leq n < j)$ and if $t_{n+1} = G(z_1,\dots,t_n,\dots,z_{n_G-1})$, then we put $u_{j+m+1} = G(z_1,\dots,u_{j+m},\dots,z_{n_G-1})$. Thus $u_{j+m} \notin \Phi(J'')$ for all m, a contradiction with Proposition 4.

Therefore card $V \leq r - q$ and we put $u = t_n$, where n is the smallest integer such that $t_n \in \Phi(J'')$. Hence it is easy to see that $U_{p+r-q} = J_{p+r-q}$ and $S_{p+r-q} \subseteq \Phi(J'')$. By the proof of Proposition 4 $\{t \in W_\Delta ;\ \lambda'(t) \geq k^{p+r-(q+1)}\} \subseteq \Phi(J'')$.

Theorem 2. Let J be a finite irreducible set of terms of a finite type Δ. Let $s = \max\{\lambda'(t);\ t \in J\}$, $k = \max\{n_F;\ F \in \Delta\} + 2$, $p = \max\{o(t);\ t \in J''\}$, $r = \operatorname{card} U_p$, $q = \operatorname{card} J_p$. Then the following conditions are equivalent.

1) Z_J is locally finite.

2) $Z_{J'}$ is locally finite.

3) $Z_{J''}$ is locally finite.

4) The algebra $W_1^{J''}$ is finite.

5) The algebra W_s^J is finite.

6) There exists an $n \leq k^{p+r-(q+1)}$ such that $\{t \in W_\Delta;\ \lambda'(t) \geq n\} \subseteq \Phi(J'')$.

7) Z_J is generated by a finite algebra.

Proof. 1) $\Longrightarrow$ 6) $\Longrightarrow$ 7) $\Longrightarrow$ 1). Apply Propositions 6 and 2.

3) $\Longleftrightarrow$ 4). Follows from Proposition 3.

3) $\Longrightarrow$ 2) $\Longrightarrow$ 1). Trivial.

1) $\Longrightarrow$ 3). By Proposition 4 there exists an positive integer m such that $S_m \subseteq \Phi(J'')$. Hence $\{t \in W_\Delta;\ \lambda'(t) \geq k^{m-1}\} \subseteq \Phi(J'')$ and consequently $Z_{J''}$ is locally finite.

1) $\Longleftrightarrow$ 5). Follows from the proof of Proposition 4.

Remark 1. For every finite irreducible set J of terms of a finite type Δ we have an algorithm to decide whether the variety Z_J is locally finite. By Proposition 6 it suffices to decide whether $U_{p+r-q} = J_{p+r-q}$, where $p = \max\{o(t);\ t \in J''\}$, $r = \operatorname{card} U_p$ and $q = \operatorname{card} J_p$. This process is obvious from the proof of this Proposition.

Remark 2. We know that under the assumptions of Theorem 2 the finiteness of W_s^J implies the local finiteness of

Z_J. If we put $h = \max\{\mathrm{card}\,(\mathrm{var}\ t);\ t \in J\}$, then it is not true in general that the finiteness of W_h^J implies the local finiteness of Z_J.

For example, let $\Delta = \{F\}$, where F is a binary operation symbol and let o denote the corresponding operation on W_Δ. Let L denote the set of all terms $t \in W_\Delta$ of the form $t = (x_{i_1} \circ x_{i_2}) \circ (x_{i_3} \circ x_{i_4})$ or $t = (x_{i_1} \circ x_{i_2}) \circ x_{i_3}$ or $t = x_{i_1} \circ (x_{i_2} \circ x_{i_3})$, where $i_1, i_2, i_3, i_4 \in \{1,2\}$. Then there exists an irreducible subset $J \subseteq L$ such that $\Phi(J) = \Phi(L)$; we have $h = 2$. It is not difficult to prove (by induction on $\lambda'(t)$) that $\{t \in W_2;\ \lambda'(t) \geq 4\} \subseteq \Phi(J)$ and consequently W_2^J is finite. However by Theorem 2 the variety Z_J is not locally finite, since $J' = J'' = \emptyset$.

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