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ON LOCAL LIPSCHITZ CONTINUITY OF SUPERPOSITION OPERATORS

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Abstract: It is proved: A superposition operator from L_p into L_q that is Lipschitzian on a fixed ball is Lipschitzian on every other ball. Furthermore, the Lipschitz constants are given. Finally the result is applied to the generalized Dirichlet problem for nonlinear partial differential equations.

Key words: Superposition operators, Lipschitz-continuity, Dirichlet problem.

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This paper deals with a local Lipschitz continuity of superposition mappings between Lebesgue spaces.
First we shall say what we mean by local Lipschitz continuity and compare this with the property L(s,t) in R. Kluge and the author [5,6] . Then a necessary and sufficient condition for local Lipschitz continuity of superposition mappings between Lebesgue spaces will be given and the corresponding Lipschitz function calculated. Finally we shall apply some of the results to the generalized Dirichlet problem for nonlinear partial differential equations. See also the concluding remarks at the end.

The results contained in this paper were first presented by the author at the Summer School on "Nonlinear Analysis and Mechanics",September 1974, Stará Lesná near Poprad,Slovakia.

1. Let X and Y be linear normed spaces and A a (nonlinear) mapping from X into Y. Then we call A locally Lipschitz continuous (l.L.) if there exists a nonnegative function $L(s,t)$ of the nonnegative arguments s, t with the properties $L(s',t') \geq L(s,t)$ if $s' \geq s$ and $t' \geq t$ and $|Ax - Ay| \leq L(|x|, |y|)\ |x - y|$ for all x, y in the domain of A.

In [5,6] R. Kluge and the author defined a local Lipschitz continuity in the case $X = Y = H$ (Hilbert space) of the form

$$|Ax - Ay| \leq L_1(|x|, |x - y|)\ |x - y|$$

with the monotone nonnegative function L_1. Obviously one definition can easily be transformed into the other. E.g., let

$$|Ax - Ay| \leq L(|x|, |y|)\ |x - y|$$

then we define $L_1(s,t) = L(s,s + t)$ and it holds

$$|Ax - Ay| \leq L(|x|, |y|)\ |x - y| \leq L(|x|, |x| + |x - y|)\ |x - y| = L_1(|x|, |x - y|)\ |x - y| .$$

Let B_1 and B_2 be Banach spaces, p and q real numbers greater than or equal to 1, $G \subseteq R^n$ a domain,

$$L_p = L_p(G,B_1), \qquad L_q = L_q(G,B_2).$$

We call an operator A mapping the whole space L_p into L_q a superposition operator (s. operator) if for every $t \in G$ there is an operator $A(t)$ of all of B_1 into B_2 with the property

$$(Ax)\ (t) = A(t)\ x(t).$$

We suppose that the family $A(t)$ fulfils the Carathéodory condition.

We denote the norms in B_1, B_2, L_p, L_q resp. by $|\ |_1$, $|\ |_2$, $\|\ \|_p$, $\|\ \|_q$. The indices will be omitted for no confusion is possible.

2. In this section the symbol A always denotes a superposition operator from L_p into L_q.
We are going to prove: If A is Lipschitz continuous on a fixed ball around the origin, then A is l.L. on the whole space. We shall give also the form of the function L(s,t).

The following Lemma will be useful for the proofs.

<u>Lemma</u>. Let $a, r_1, \dots, r_N$ be nonnegative real numbers.

(i) If $a \geq 1$, then

$$\sum r_i^a \leq (\sum r_i)^a \leq N^{a-1} \sum r_i^a.$$

(ii) If $a \leq 1$, then

$$\sum r_i^a \geq (\sum r_i)^a \geq N^{a-1} \sum r_i^a.$$

(Summation runs over $i = 1, \dots, N$).

For our investigations we distinguish the two cases $p \leq q$ and $p > q$.

<u>Theorem 1</u>. Let $p \leq q$, $a \geq p/q$ be a real number and

(1) $$\| Ax - Ay \| \leq M \| x - y \|^a$$

for all x, y from a fixed ball with radius $r > 0$ around the origin of L_p. Then (1) holds for every x and $y \in L_p$.

Remark. $a > 1$ means that A is constant. In the following we exclude this non interesting case by the restriction $a \leq 1$.

Proof of Theorem 1. Let $x, y \in L_p$. We decompose the domain G into measurable sets G_i such that

$\bigcup_i G_i = G$, $G_i \cap G_j = \emptyset$ (empty set) for $i \neq j$ and

$$\left(\int_{G_i} |x(t)|^p dt\right)^{1/p} \leq r, \quad \left(\int_{G_i} |y(t)|^p dt\right)^{1/p} \leq r.$$

We define $x_i, y_i \in L_p$ as

$$x_i(t) = \begin{cases} x(t), & t \in G_i \\ 0, & t \notin G_i \end{cases} \qquad y_i(t) = \begin{cases} y(t), & t \in G_i \\ 0, & t \notin G_i \end{cases}$$

Then $\|x_i\| \leq r$, $\|y_i\| \leq r$, and it holds

$$\|Ax - Ay\|^q = \int_G |A(t)x(t) - A(t)y(t)|^q dt =$$

$$= \sum_i \int_{G_i} |A(t)x(t) - A(t)y(t)|^q dt$$

$$= \sum_i \|Ax_i - Ay_i\|^q \leq \sum_i M^q \|x_i - y_i\|^{q.a}$$

$$= M^q \sum_i \left(\int_G |x_i(t) - y_i(t)|^p\right)^{q.a/p}$$

$q.a/p \geq 1$ and the Lemma implies

$$\|Ax - Ay\|^q \leq M^q \left(\sum_i \int_G |x_i(t) - y_i(t)|^p\right)^{q.a/p} =$$

$$= M^q \left(\int_G |x(t) - y(t)|^p\right)^{q.a/p},$$

this means

$$\|Ax - Ay\| \leq M \|x - y\|^a, \qquad \text{q.e.d.}$$

<u>Corollary</u>. In the case $p \leq q$ every l.L.s. mapping is Lipschitz continuous.

Proof. Put $a = 1$. It is obvious that an l.L.s. mapping fulfils the assumption of Theorem 1.

<u>Theorem 2</u>. Let $p > q$ and

$$\|Ax - Ay\| \leq M \|x - y\| \tag{2}$$

for all x, y from a fixed ball with radius $r > 0$ around the origin of L_p. Then

(3) $$\| Ax - Ay \| \leq [c_1(\| x \|^{(p-q)/q} + \| y \|^{(p-q)/q}) + c_2] \| x - y \|$$

with $c_1 = M/r^{(p-q)/q}$, $c_2 = M$, holds for every $x, y \in L_p$.

Proof. Let $x, y \in L_p$ and

$$\| x \|^p = m\, r^p + \varepsilon_1, \ m \geq 0, \text{ integer}, \ 0 \leq \varepsilon_1 < r^p$$

$$\| y \|^p = n\, r^p + \varepsilon_2, \ n \geq 0, \text{ integer}, \ 0 \leq \varepsilon_2 < r^p.$$

We decompose the domain G into measurable sets G_i, $i = 0, \ldots, m$, such that

$$\bigcup_i G_i = G, \ G_i \cap G_j = \emptyset \text{ for } i \neq j \text{ and}$$

$$\int_{G_i} | x(t) |^p \, dt = r^p, \ i = 1, \ldots, m, \ \int_{G_0} | x(t) |^p \, dt = \varepsilon_1.$$

Let further

$$\int_{G_i} | y(t) |^p \, dt = n_i\, r^p + \varepsilon'_i, \ n_i \geq 0 \text{ integer}, \ 0 \leq \varepsilon'_i < r^p,$$

for $i = 0, \ldots, m$.

Then $\sum_{i=0}^{m} n_i \leq n$.

Now we decompose the sets G_i, $i = 0, \ldots, m$, in the same way as before into measurable sets G_{ij}, $j = 0, \ldots, n_i$ such that

$$\int_{G_{ij}} | y(t) |^p \, dt = r^p, \ j = 1, \ldots, n_i, \ \int_{G_{i0}} | y(t) |^p \, dt = \varepsilon'_i.$$

Then

$$\left(\int_{G_{ij}} | x(t) |^p \, dt \right)^{1/p} \leq r, \ \left(\int_{G_{ij}} | y(t) |^p \, dt \right)^{1/p} \leq r.$$

We estimate the number N of the G_{ij}. It holds

$$(\|x\|^p + \|y\|^p)/r^p = [(m+n)r^p + \varepsilon_1 + \varepsilon_2]/r^p \geq m + n,$$

therefore

$$N \leq \sum_{i=0}^{m} (n_i + 1) \leq n + m + 1 \leq (\|x\|^p + \|y\|^p)/r^p + 1.$$

We define

$$x_{ij}(t) = \begin{cases} x(t), & t \in G_{ij} \\ 0, & t \notin G_{ij} \end{cases} \qquad y_{ij}(t) = \begin{cases} y(t), & t \in G_{ij} \\ 0, & t \notin G_{ij} \end{cases}$$

Then $x_{ij}, y_{ij} \in L_p$ and $\|x_{ij}\| \leq r$, $\|y_{ij}\| \leq r$.

We estimate

$$\begin{aligned}
\|Ax - Ay\|^q &= \int_G |A(t)x(t) - A(t)y(t)|^q \, dt = \\
&= \sum_{i,j} \int_{G_{ij}} |A(t)x(t) - A(t)y(t)|^q \, dt \\
&= \sum_{i,j} \int_G |A(t)x_{ij}(t) - A(t)y_{ij}(t)|^q \, dt = \\
&= \sum_{i,j} \|Ax_{ij} - Ay_{ij}\|^q \leq M^q \sum_{i,j} \|x_{ij} - y_{ij}\|^q = \\
&= M^q \sum_{i,j} \left(\int_G |x_{ij}(t) - y_{ij}(t)|^p \, dt\right)^{q/p},
\end{aligned}$$

$q/p < 1$, therefore the Lemma gives

$$\leq M^q \, . \, N^{1-q/p} \left(\sum_{i,j} \int_{G_{ij}} |x(t) - y(t)|^p \, dt\right)^{q/p}$$

$$= M^q \, . \, N^{(p-q)/p} \|x - y\|^q,$$

that means

$$\|Ax - Ay\| \leq M \left[\|x\|^p/r^p + \|y\|^p/r^p + 1\right]^{(p-q)/pq} \|x - y\| .$$

It holds

$$0 < (p - q)/pq = \frac{1}{q} - \frac{1}{p} < 1$$

and our assertions follows by the Lemma. Q.e.d.

3. Here we assume that A is l.L.s. and ask for $A(t)$. Again we consider first the case $p \leq q$.

Theorem 3. Let $p \leq q$ and

$\| Ax - Ay \| \leq M \| x - y \|^a$, $p/q \leq a \leq 1$, a fixed, for all $x, y \in L_p$.

Let G be of finite measure g. Then

$$| A(t)u - A(t)v | \leq M g^b | u - v |^a, \quad b = (aq - p)/pq,$$

for all $u, v \in B_1$ and almost all $t \in G$.

Proof. Let $u, v \in B_1$ and

(4) $\quad | A(t)u - A(t)v | > M g^b | u - v |^a$ for $t \in \Delta \subseteq G$.

Δ is a measurable set. We define

$$x(t) = \begin{cases} u, & t \in \Delta \\ o, & t \notin \Delta \end{cases}, \qquad y(t) = \begin{cases} v, & t \in \Delta \\ o, & t \notin \Delta \end{cases}$$

Then $x, y \in L_p$ and

$$\| Ax - Ay \| = \left(\int_G | A(t)x(t) - A(t)y(t) |^q \, dt \right)^{1/q}$$

$$= \left(\int_\Delta | A(t)u - A(t)v |^q \, dt \right)^{1/q}$$

$$\leq M \| x - y \|^a = M \left(\int_G | x(t) - y(t) |^p \, dt \right)^{a/p}$$

(5) $$= M \left(\int_\Delta | u - v |^p \, dt \right)^{a/p} = M(\text{mes } \Delta)^{a/p} | u - v |^a .$$

Otherwise, if we suppose mes $\Delta > 0$ we get from (4)

$$\left(\int_\Delta |A(t)u - A(t)v|^q \, dt\right)^{1/q} > M\, g^b (\text{mes}\, \Delta)^{1/q} |u - v|^a .$$

This gives together with (5)

$$M\, g^b (\text{mes}\, \Delta)^{1/q} < M (\text{mes}\, \Delta)^{a/p} \quad \text{or}$$

$$g^b < (\text{mes}\, \Delta)^b .$$

Because of $aq - p \geq 0$ this contradicts $g \geq \text{mes}\, \Delta$. It follows mes $\Delta = 0$, q.e.d.

Let now be $p > q$. We first consider the special case where $|A(t)u - A(t)v|$ does not depend on t, we call the corresponding A independent.

<u>Theorem 4</u>. Let $p > q$, A independent and l.L. superposition operator.

Then

(6) $$|A(t)u - A(t)v| \leq [d_1(|u|^{(p-q)/q} + |v|^{(p-q)/q}) + d_2]\, |u - v|$$

for all $u, v \in B_1$. Here $d_1 = c_1$, $d_2 = g^{(q-p)/pq} c_2$, where c_1, c_2 are the constants of Theorem 2 and $g = \text{mes}\, G$.

Proof. Let $u, v \in B_1$. From Theorem 2 we get

$$\begin{aligned} \|Au - Av\| &= \left(\int_G |A(t)u - A(t)v|^q \, dt\right)^{1/q} = \\ &= g^{1/q} |A(t)u - A(t)v| \\ &\leq [c_1(\|u\|^{(p-q)/q} + \|v\|^{(p-q)/q}) + c_2]\, \|u - v\| \\ &= [c_1\, g^{(p-q)/pq}(|u|^{(p-q)/q} + \\ &+ |v|^{(p-q)/q}) + c_2]\, g^{1/p} |u - v| . \end{aligned}$$

Q.e.d.

Corollary. Let $p > q$ and A an independent and l.L. superposition operator. Then

$$|A(t)u| \leq g(t) + c\,|u|^{p/q}, \quad g(t) \in L_q(G,R_1).$$

Proof. Theorem 4 gives for $v = 0$

$$|A(t)u| \leq |A(t)0| + (d_1 |u|^{(p-q)/q} + d_2)\,|u|$$

$$= |A(t)0| + d_1 |u|^{p/q} + d_2 |u|$$

$$\leq |A(t)0| + d_2 + (d_1 + d_2)\,|u|^{p/q}, \qquad \text{q.e.d.}$$

The Corollary is the Lemma of Krasnoselski [8] in our special case. If we assume (6) to hold together with the Carathéodory condition for a family A(t) of operators then the superposition operator A will be an l.L. operator from L_p into L_q with Lipschitz function as in Theorem 2. This can be shown easily by means of the Hölder inequality. So (6) is a necessary and sufficient condition for the independent superposition operator A to be an l.L. mapping from L_p into L_q.

As an Example we consider in $R_1 = B_1 = B_2$ for integer m, n, $1 \leq m \leq n$ the mapping

$$A(t)x = r\,x^n + s\,x^m, \quad r, s \text{ real numbers.}$$

We have

$$|A(t)x - A(t)y| = |r(x^n - y^n) + s(x^m - y^m)|$$

$$\leq |r|\,|x^n - y^n| + |s|\,|x^m - y^m|$$

$$\leq [\,|r|\,(|x| + |y|)^{n-1} + |s|\,(|x| + |y|)^{m-1}]\,|x - y|$$

$$\leq [\,(|r| + |s|)\,(|x| + |y|)^{n-1} + |s|\,]\,|x - y|$$

$$\leq [2^{n-2}(|r| + |s|)(|x|^{n-1} + |y|^{n-1}) + |s|]\,|x - y|.$$

Thus A is an l.L.s. mapping of L_p into $L_{p/n}$. Especially it can be seen from the example that the term c_2 in (3) resp. d_2 in (6) cannot be omitted in general.

In the case where the independence of A is not assumed we have the

Theorem 5. Let $p > q$ and A an l.L.s. mapping. Let further be k_1, k_2 nonnegative (universal) constants and for $u, v \in B_1$

$$(7)\quad |A(t)u - A(t)v| > [k_1(|u|^h + |v|^h) + k_2]\,|u - v|,$$

$$h = (p - q)/q,$$

on the (measurable) set $\Delta = \Delta(u,v)$ with mes $\Delta > 0$. Then it holds

$$(8)\quad (k_1 - c_1)(|u|^h + |v|^h) + k_2 < c_2 (\text{mes}\, \Delta)^{-(1/q-1/p)},$$

where c_1, c_2 are the constants of Theorem 2.

Proof. Let x and y be the elements of L_p defined by

$$x(t) = \begin{cases} u, & t \in \Delta \\ o, & t \notin \Delta \end{cases}, \qquad y(t) = \begin{cases} v, & t \in \Delta \\ o, & t \notin \Delta \end{cases},$$

then

$$\|x\| = \left(\int_G |x(t)|^p\, dt\right)^{1/p} = \left(\int_\Delta |u|^p\, dt\right)^{1/p} =$$

$$= (\text{mes}\, \Delta)^{1/p}\, |u|,$$

$$\|y\| = (\text{mes}\, \Delta)^{1/p}\, |v|, \quad \|x - y\| = (\text{mes}\, \Delta)^{1/p}\, |u - v|$$

and

$$\| Ax - Ay \| = \left(\int_{\Delta} | A(t)u - A(t)v|^q \, dt \right)^{1/q} \leq$$

$$\leq [c_1(\| x \|^h + \| y \|^h) + c_2] \, \| x - y \|$$

$$= [c_1 (\text{mes } \Delta)^{h/p} (|u|^h + |v|^h) + \qquad (9)$$

$$+ c_2] (\text{mes } \Delta)^{1/p} |u - v| .$$

Otherwise (7) gives by integration

$$\left(\int_{\Delta} | A(t)u - A(t)v|^q \, dt \right)^{1/q} > (\text{mes } \Delta)^{1/q} [k_1(|u|^h +$$

$$+ |v|^h) + k_2] \, |u - v| .$$

This gives together with (9)

$$k_1(|u|^h + |v|^h) + k_2 < c_1(|u|^h + |v|^h) +$$

$$+ c_2 (\text{mes } \Delta)^{1/p - 1/q}, \qquad \text{q.e.d.}$$

Consequences of Theorem 5:

(i) In the case $c_2 = 0$ we get a contradiction if we take $k_1 = c_1$, $k_2 = 0$. That means mes $\Delta = 0$.

(ii) If we take $k_1 = c_1$, $k_2 = c_2 k$, $c_2 \neq 0$ implies $(\text{mes } \Delta)^{1/q - 1/p} < 1/k$.

This means that we can make mes Δ arbitrarily small if we choose k large enough.

(iii) If we take $k_1 > c_1$, k_2 fixed, then mes $\Delta \longrightarrow 0$ for $|u| + |v| \longrightarrow \infty$.

4. As an application we consider here the generalized nonlinear Dirichlet problem in $L_2(G)$

$$a(u,v) = \sum_{|\alpha| \le 1} (A_\alpha(x,t(u)),\ D^\alpha v) = (\bar{g},v),$$

where $u,v \in W_2^1(G)$, G a domain of R^n (with the usual suppositions), D^α the operator of generalized derivation, α a multi-index and $t(u)$ the vector of the generalized derivatives of u. It is known that under certain assumptions on A_α the problem is equivalent to an operator equation

$$Tu = g$$

in H_1, the closure of $\dot{C}^\infty$ in W_2^1. These assumptions are

(i) Carathéodory condition,

(ii) growth condition.

Besides, usually for the existence proof a

(iii) monotonicity and coercivity condition

and for numerical solution (iteration methods) a

(iv) Lipschitz condition

are assumed. (See e.g. [1,4] .)

Motivated by Theorem 4 and the Corollary and Remark following on it we replace the conditions (ii) and (iv) by a local Lipschitz condition of the form

$$\text{(v)} \quad |A_\alpha(x,t) - A_\alpha(x,t')| \le \sum_{|\beta| \le 1} L_{\beta\alpha}(|t_\beta|, |t'_\beta|)\, |t_\beta - t'_\beta|,$$

where t is the vector of the real numbers t_β and

$$\text{(10)} \quad L_{\beta\alpha}(s,s_1) = \begin{cases} \text{arbitrary (monotone) for } |\beta| \le 1 - n/2, \\ \qquad\qquad \alpha \text{ arbitrary} \\ c_{\beta\alpha}\left(s^{r_\beta - 1} + s_1^{r_\beta - 1}\right) + d_{\beta\alpha}, \quad |\alpha| < 1 - n/2 \\ c_{\beta\alpha}\left(s^{\frac{\varkappa_\beta(\varkappa_\alpha - 1)}{\varkappa_\alpha} - 1} + s_1^{\frac{\varkappa_\beta(\varkappa_\alpha - 1)}{\varkappa_\alpha} - 1}\right) + d_{\beta\alpha} \\ \qquad\qquad 1 - n/2 \le |\alpha| \le 1. \end{cases}$$

Here $c_{\beta\alpha}$ and $d_{\beta\alpha}$ are arbitrary nonnegative constants and the r_β are defined by the

Sobolev Imbedding Theorem. Let $u \in W_2^1(G)$, then for $1 - n/2 \leq |\beta| \leq 1$ it holds $D^\beta u \in L_{r_\beta}$, $r_\beta = 2n/[n - 2(1 - |\beta|)]$,

and $|D^\beta u|_{L_{r_\beta}} \leq M_\beta \| u \|_1$;

for $|\beta| < 1 - n/2$ it holds $D^\beta u \in C$ and $|D^\beta u|_C \leq M_\beta \| u \|_1$.

It can be easily seen that $L_{\beta\alpha}$ is constant if and only if $|\alpha| = |\beta| = 1$. In all other cases our condition (v) is weaker than any condition (iv). Adding 1 to the exponents in the right hand side of (10) we get exactly the exponents ocurring usually in condition (ii). Thus the growth condition resulting from condition (v) (in the same way as it is done in the Corollary to Theorem 4) is just the usual growth condition (ii). This shows that condition (v) is quite natural. Furthermore it is obvious by Theorems 2 and 4 that within the frame of the existing theory condition (v) cannot be weakened any more.

Finally we show that condition (v) leads to an l.L. operator T. Thus the methods developed in [3,4] can be used for a numerical solution of the problem Tu = g.

Theorem 6. Under the conditions (i) and (v) The Dirichlet problem is equivalent to the operator equation Tu = g in H_1, and the operator T is l.L. with the Lipschitz function

$$L(\|u\|_1, \|v\|_1) = \sum_{\alpha,\beta} m_{\beta\alpha} L_{\beta\alpha}(M_\beta \|u\|_1, M_\beta \|v\|_1),$$

where $m_{\beta\alpha}$ are certain nonnegative constants and M_β are the same as in the Sobolev imbedding theorem cited above.

Proof. We mentioned above that (v) implies the usual growth condition (ii). So we can proceed first in the same way as for example in [1,2] . For the remaining l.L. property of T we have to estimate

$$|(Tu - Tv,h)_1| = |\sum_{|\alpha|\le 1} \int_G (A_\alpha(x,t(u)) - A_\alpha(x,t(v)) D^\alpha h\, dx|$$

$$\le \sum_{\beta,\alpha} \int_G L_{\beta\alpha}(|D^\beta u|, |D^\beta v|)|D^\beta(u - v)| \, |D^\alpha h|\, dx$$

$$\le L(\|u\|_1, \|v\|_1)\|u - v\|_1 \; \|h\|_1$$

using the Sobolev imbedding theorem and the Hölder inequality in the form

$$\int_G |x||y||z|\, dt \le (\int_G |x|^{p_1} dt)^{1/p_1} (\int_G |y|^{p_2} dt)^{1/p_2} (\int_G |z|^{p_3} dt)^{1/p_3},$$

where $p_i \ge 1$, $1/p_1 + 1/p_2 + 1/p_3 = 1$.

Concluding remarks. The following assertion is due to M.M. Vainberg ([9],[10]):

An s. operator that is uniformly continuous on a fixed ball in L_p has the same property on every other ball in L_p.

As to Lipschitz continuity our Theorems 1 and 2 are of the same type: An s. operator that is Lipschitzian on a fixed ball in L_p is Lipschitzian on every other ball in L_p. Furthermore the Lipschitz function is given as

$$(11)\quad L(\|x\|,\|y\|) = M\left[\left(\frac{\|x\|}{r}\right)^{(p/q)-1} + \left(\frac{\|y\|}{r}\right)^{(p/q)-1} + 1\right],\quad p \geq q,$$

(see Theorem 2). The stronger result

$$L(\|x\|,\|y\|) = M \cdot \max\left\{\left(\frac{\|x\|}{r}\right)^{(p/q)-1}, \left(\frac{\|y\|}{r}\right)^{(p/q)-1}, 1\right\}$$

is due to W. Müller ([7]).

Most of our reflections about Lipschitz continuity can without difficulty be transferred to Hölder continuity and strong monotonicity ([2]). E.g., generalizing (11) the Hölder function corresponding to the Hölder exponent $a \leq p/q$ is

$$H(\|x\|,\|y\|) = M\left[\left(\frac{\|x\|}{r}\right)^{(p/q)-a} + \left(\frac{\|y\|}{r}\right)^{(p/q)-a} + 1\right].$$

For related results see also J. Daneš [3].

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