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FINE TOPOLOGIES AS EXAMPLES OF NON-BLUMBERG Baire SPACES

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Abstract: Any \( \mathfrak{B} \)-harmonic space with countable base in axiomatic potential theory in which the points are polar endowed with the fine topology is non-Blumberg Baire space provided the continuum hypothesis is assumed.

Key words: Blumberg space, Baire space, fine topology in potential theory, density topology.

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In 1922, H. Blumberg [2] showed that for any real function \( f \) defined on the real line \( \mathbb{R} \), there is a dense subset \( D \) of \( \mathbb{R} \) such that the restriction of \( f \) to \( D \) is continuous. We shall say that a topological space \( X \) is a Blumberg space if for any real function \( f \) on \( X \), there is a dense subset \( D \) of \( X \) such that \( f/D \) is continuous. The result of J.C. Bradford and C. Goffman 1960 [3] shows that for a metric space, \( X \) is Blumberg if and only if \( X \) is a Baire space. While any topological Blumberg space is Baire, the converse is not true in general. The first examples of non-Blumberg Baire space are due to Jr. H.E. White 1974 [9] (assuming the continuum hypothesis, the density topology on the real line serves an example) and 1975 [10] (e.g., any Baire space of cardinality, weight and density character \( 2^{\aleph_0} \) satisfying

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the countable chain condition, in which sets of the first
category and nowhere dense sets coincide), to R. Levy 1973
[6] (any \( \eta_1 \)-set of cardinality \( 2^{\aleph_0} \) ) and 1974 [7], and
to W.A.R. Weiss 1975 [8] (even an example of compact non-
Blumberg space). See also [11], where more detail discuss-
ions and interesting results can be found.

Using certain elementary theorem, we will give furth-
er examples of non-Blumberg Baire spaces. In particular,
any abstract harmonic space equipped with the fine topolo-
gy sets such an example.

**Notation.** Given any topological space, \( b(A) \) will de-
note the derived set of \( A \).

**Theorem 1.** Let \( X \) be a topological space without iso-
lated points such that any dense subset of \( X \) is of car-
dinality \( 2^{\aleph_0} \). If the cardinality of the system \( \{ b(A); A \subseteq X \} \) is less or equal to \( 2^{\aleph_0} \), then \( X \) is not a Blumberg
space.

**Proof.** For any dense subset \( A \) of \( X \), and for any real
function \( f \) on \( X \) we put
\[
\tilde{f}_A(y) = \lim_{x \to y} \sup_x f(x) = \sup \{ a; y \in b(x); f(x) \leq a \}, \quad y \in X.
\]
Since we always have
\[
\{ y \in X; \tilde{f}_A(y) \leq a \} = \bigcap_{n \in \mathbb{Q}} \{ y \in A; f(y) \leq a \},
\]
it follows that any \( \tilde{f}_A \) is measurable with respect to cer-
tain system of sets of cardinality \( \leq 2^{\aleph_0} \). By this obser-
vation one reaches the conclusion that the system
\[
\tilde{f}_A : = \{ \tilde{f}_A; A \text{ is dense in } X, f \text{ is a function on } X \}
\]
is of cardinality $\leq 2^{\aleph_0}$. Let $\Omega$ be the first ordinal number of cardinality $2^{\aleph_0}$. Suppose now that \( \{ x_\alpha \}_{\alpha < \omega} \) is the set of all points of $X$, and \( \{ g_\alpha \}_{\alpha < \omega} \ (\omega \leq \Omega) \) is the set of all functions from $\Phi$. By transfinite induction we can construct a function $f$ on $X$ such that

$$f(x_\alpha) = g_\gamma(x_\alpha) \text{ for any } \gamma < \alpha \ (\gamma < \omega).$$

Then, for any $g \in \Phi$, the cardinality of \( \{ x \in X; f(x) = g(x) \} \) is less than $2^{\aleph_0}$. Hence, it follows easily that there is no dense subset $A$ of $X$ for which $f \upharpoonright A$ is continuous. If it existed, so $f \upharpoonright A \in \Phi$, and this is a contradiction since $f = f \upharpoonright A$ on $A$ and cardinality of $A$ is $2^{\aleph_0}$.

**Fine topologies in potential theory.** Assume that $X$ is a $\aleph_0$-harmonic-space with countable base in the sense of axiomatics C.Constantinescu and A.Cornea [4]. By this we mean a locally compact topological space with countable base (therefore, $X$ is a metric separable space) which is endowed with a hyperharmonic sheaf and satisfies certain axioms. The fine topology on $X$ is the coarsest topology on $X$ which is finer than the initial topology and in which any hyperharmonic function on $X$ is continuous. It is known that there are not isolated points in the fine topology ([4], Corollary 5.1.2), and that $X$ endowed with the fine topology is a Baire space ([4], Corollary 5.1.1). Moreover, if we shall suppose that the points of $X$ are polar, then the derived set $b(A)$ of any subset $A \subseteq X$ in the fine topology is exactly the set of all points of $X$ where $A$ is not thin ([4], Exercise 7.2.1). Therefore, $b(A)$ is always of type $G_\tau$ in the initial topo-
logy ([4], Corollary 7.2.1), and thus the cardinality of the system \( \{b(A); A \in X\} \) is less or equal to \( 2^{\aleph_0} \). Further, the whole space \( X \) is uncountable ([4], Exercise 5.1.5), and any countable subset of \( X \) is polar. Hence, it is always closed in the fine topology. Thus, assuming the continuum hypothesis, any dense subset of \( X \) must be of cardinality \( 2^{\aleph_0} \).

Applying our theorem, we get the following important examples of non-Blumberg Baire spaces.

**Theorem 2.** Assuming the continuum hypothesis, any abstract \( \mathcal{B} \)-harmonic space with a countable base endowed with the fine topology, in which every point is polar, is a non-Blumberg Baire space.

**Remark.** The same theorem remains true if we suppose that the points of \( X \) are semi-polar only and axiom of thinness ( = any semi-polar set is finely closed) is satisfied. In both cases, we can also replace the continuum hypothesis with the assumption that any subset of \( X \) of cardinality \( < 2^{\aleph_0} \) is semi-polar. (It is sufficient to use the facts that, in the fine topology, any semi-polar set is of the first category and the whole space \( X \) is of the second category in itself.)

**Density topology.** Consider now the ordinary density topology in the Euclidean space \( \mathbb{R}^n \) introduced by C. Goffman and D. Waterman 1961 in [5]. In this topology \( \mathbb{R}^n \) is a Baire space without isolated points. Moreover, any derived set in density topology is of type \( G_{\delta\sigma} \) in the Euclidean topology.
Thus, the theorem 1 gives again the following result which is due to Jr. H.E. White.

Theorem 3. If any subset of $\mathbb{R}^n$ of cardinality $< 2^{2^{<\omega}}$ has a Lebesgue measure zero, then $\mathbb{R}^n$ endowed with the density topology is a Baire non-Blumberg space.

References


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