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ON THE STRUCTURE OF FIXED POINT SETS OF PSEUDO-CONTRACTIVE  
MAPPINGS

Rainald SCHÖNEBERG, Aachen

**Abstract:** Let  $(E, \| \cdot \|)$  be a Banach-space,  $X$  a closed and bounded subset of  $E$  and let  $f: X \rightarrow E$  be a pseudo-contractive mapping. It is shown that under certain conditions the set  $\text{Fix}(f)$  of fixed points of  $f$  is metrically convex and hence pathwise connected.

**Key words:** Inward, nonexpansive, pseudo-contractive,  $k$ -set-contraction, metrically convex, pathwise connected.

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The purpose of this note is to give some conditions which assure that the fixed point set of a pseudo-contractive mapping is metrically convex and hence pathwise connected. A recent result of the author is basic for the proofs.

**Definition 1.** Let  $(E, \| \cdot \|)$  be a Banach-space and  $X \subset E$ .  $X$  is said to be metrically convex:  $\iff$

$$: \iff \forall \substack{x, y \in X \\ x \neq y} \quad \exists \substack{z \in X \\ z \neq x \wedge z \neq y} \wedge \|x - y\| = \|x - z\| + \|y - z\|$$

**Remark 1.** Every convex set is metrically convex but the converse isn't true in general ( $E := \mathbb{R}^2$ ,  $\| \cdot \| := \text{max-norm}$ ,  $X := \{ (|t|, t) \mid t \in [-1, 1] \}$ ).

A fundamental property of a metrically convex set is described by

Proposition 1 (K. Menger). Let  $(E, \| \cdot \|)$  be a Banach-space,  $X \subset E$  be closed and metrically convex,  $x, y \in X$  and  $d := \|x - y\|$ .

Then there is  $\varphi : [0, d] \rightarrow X$  such that

- (i)  $\varphi(0) = x \wedge \varphi(d) = y$
- (ii)  $\forall a, b \in [0, d] \quad \|\varphi(a) - \varphi(b)\| = |a - b|$

(i.e.  $\varphi$  is an isometry)

Proof see [1], Theorem 14.1.

Corollary 1. Let  $(E, \| \cdot \|)$  be a Banach-space and let  $X \subset E$  be a closed and metrically convex subset of  $E$ .

Then  $X$  is pathwise connected.

Proof: Obvious.

Corollary 2. Let  $(E, \| \cdot \|)$  be a strictly convex Banach-space and let  $X \subset E$  be closed.

Then  $X$  is convex if and only if  $X$  is metrically convex.

Proof. If  $X$  is convex then  $X$  is obviously metrically convex. Conversely suppose  $X$  is metrically convex and let  $x, y \in X$ . By Proposition 1 there is an isometry

$\varphi : [0, \|x - y\|] \rightarrow X$  such that  $\varphi(0) = x$  and

$\varphi(\|x - y\|) = y$ . Since  $(E, \| \cdot \|)$  is strictly convex,  $\varphi$  is affine (see [9]) and hence  $\varphi[[0, \|x - y\|]]$  is convex. Therefore  $\text{co}(\{x, y\}) := \text{convex hull of } \{x, y\} \subset \varphi[[0, \|x - y\|]] \subset X$  i.e.  $X$  is convex.

Definition 2. Let  $(E, \| \cdot \|)$  be a Banach-space,  $X \subset E$  and let  $f : X \rightarrow E$ .

(1)  $f$  is said to be nonexpansive:  $\iff \forall x, y \in X \quad \|f(x) - f(y)\| \leq \|x - y\|$

(2)  $f$  is said to be pseudo-contractive :  $\iff$

$$\iff \forall_{x, y \in X} \quad \forall_{r \geq 0} \quad \|x - y\| \leq \|(1+r)(x - y) - r(f(x) - f(y))\|$$

Remark 2. Pseudo-contractive mappings are characterized by the property:  $f$  is pseudo-contractive if and only if  $\text{Id} - f$  is accretive (see [2]). It is easily seen that these mappings include the non-expansive mappings.

In [1] we proved the following theorem:

Theorem. Let  $(E, \| \cdot \|)$  be a Banach-space and suppose  $M$  is a closed subset of  $E$  such that every nonempty, closed, bounded and convex subset of  $M$  possesses the fixed point property with respect to nonexpansive selfmappings. Let  $g: M \rightarrow E$  be nonexpansive such that at least one of the following conditions holds:

- (A)  $M$  is convex and  $g[M] \subset M$
- (B)  $\text{Fix}(g) \cap \partial M = \emptyset$  <sup>1)</sup>

Then the (possibly empty) fixed point set of  $g$  is metricaly convex and hence pathwise connected.

The approach of [4], showing how fixed point theorems for pseudo-contractive mappings may be derived from the fixed point theory of nonexpansive mappings, may be modified to obtain the following two theorems:

Theorem 1. Let  $(E, \| \cdot \|)$  be a Banach-space and suppose  $X$  is a nonempty, closed, bounded and convex subset of  $E$  such that every nonempty, closed, bounded and convex subset of  $X$  possesses the fixed point property with respect to nonexpan-

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1)  $\partial M :=$  boundary of  $M$

sive selfmappings. Let  $f: X \rightarrow E$  be a  $k$ -set-contraction (in the sense of the Kuratowski-measure of noncompactness [6],  $k \geq 0$ ), pseudo-contractive and inward (i.e.

$$\forall x \in \partial X \quad \exists u \in X \quad \exists c \geq 0 \quad f(x) = x + c(u - x), \text{ see [3]}).$$

Then  $\text{Fix}(f)$  is nonempty, bounded, closed and metrically convex.

Proof. Let  $\lambda \in (0,1)$  such that  $\lambda \cdot k < 1$  and define  $T: X \rightarrow E$  by  $T(x) := x - \lambda \cdot f(x)$ . Because  $f$  is pseudo-contractive we have

$$(i) \quad \forall x, y \in X \quad \|T(x) - T(y)\| \geq (1 - \lambda) \|x - y\|$$

Let now  $y \in X$ . Defining  $h_y: X \rightarrow E$  by  $h_y(x) := \lambda f(x) + (1 - \lambda)y$  it is easily verified that  $h_y$  is condensing (because  $\lambda \cdot k < 1$ ) and inward (because  $f$  is inward). Hence by [8] there is  $x \in X$  with  $h_y(x) = x$  i.e.  $T(x) = (1 - \lambda)y$ . Thus we have shown:

$$(ii) \quad M := (1 - \lambda)X \subset T[X]$$

Because of (i) and (ii) we may define  $g: M \rightarrow M$  by  $g(x) := (1 - \lambda)T^{-1}(x)$ . Then  $g$  is nonexpansive (because of (i)) and every nonempty, closed, bounded and convex subset of  $M$  possesses the fixed point property with respect to nonexpansive selfmappings. Since  $\text{Fix}(g) = (1 - \lambda)\text{Fix}(f)$  the theorem stated above gives the assertion.

Corollary 3. Let  $(E, \| \cdot \|)$  be a Banach-space,  $\emptyset \neq X \subset E$  be closed, bounded and convex and let  $f: X \rightarrow X$  be a  $k$ -set-contraction for some  $k < 1$  and pseudo-contractive.

Then  $\text{Fix}(f)$  is nonempty, compact and pathwise connected.

Proof. Let  $C_1 := \overline{\text{co}}(f[X])$  and  $C_{n+1} := \overline{\text{co}}(f[C_n])$  for

$n \geq 1$ . Then  $C_\infty := \bigcap_{n \geq 1} C_n$  is nonempty, compact and convex such that  $f[C_\infty] \subset C_\infty$  (see e.g. [6]). Furthermore  $\text{Fix}(f) \subset C_\infty$ . Setting  $g := f|_{C_\infty}$  Theorem 1 and Corollary 1 yield - observing Schauder's fixed point theorem - that  $\text{Fix}(g)$  is nonempty, compact and pathwise connected. Because of  $\text{Fix}(f) = \text{Fix}(g)$  we are done.

Theorem 2. Let  $(E, \|\cdot\|)$  be a Banach-space such that every nonempty, closed, bounded and convex subset of  $E$  possesses the fixed point property with respect to nonexpansive selfmappings. Let furthermore  $X \subset E$  be open and bounded and let  $f: \bar{X} \rightarrow E$  be a  $k$ -set-contraction ( $k \geq 0$ ) and pseudo-contractive such that  $\text{Fix}(f) \cap \partial X = \emptyset$ .

Then the (possibly empty) fixed point set of  $f$  is closed, bounded and metrically convex.

Proof. Choose  $\lambda \in (0,1)$  such that  $\lambda \cdot k < 1$  and define  $T: \bar{X} \rightarrow E$  by  $T(x) := x - \lambda f(x)$ . Set  $M := T[\bar{X}]$ . Then  $M$  is closed because  $X$  is bounded and  $\lambda f$  is condensing. Since  $f$  is pseudo-contractive we may define  $g: M \rightarrow E$  by  $g(x) := (1 - \lambda)T^{-1}(x)$ . Then  $g$  is nonexpansive. Now Nussbaum's invariance of domain theorem [6] yields that  $T$  maps  $X$  into the interior of  $M$ . Therefore  $\partial M \subset T[\partial X]$  which implies that  $\text{Fix}(g) \cap \partial M = \emptyset$ . Observing  $\text{Fix}(f) = (1 - \lambda) \text{Fix}(f)$  we are done.

Corollary 4. Let  $(E, \langle \cdot, \cdot \rangle)$  be a Hilbert-space and let  $X \subset E$  be an open, bounded neighborhood of the origin. Let  $f: \bar{X} \rightarrow E$  be a  $k$ -set-contraction for some  $k \geq 0$  and pseudo-contractive such that

$$\forall_{x \in \partial X} \quad \forall_{\lambda \geq 0} \quad f(x) = \lambda x \implies \lambda < 1.$$

Then  $\text{Fix}(f)$  is nonempty, closed, bounded and convex.

Proof. Theorem 4 of [10], Theorem 2 and Corollary 2.

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