Marián J. Fabián
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Commentationes Mathematicae Universitatis Carolinae, Vol. 18 (1977), No. 1, 19--39

Persistent URL: http://dml.cz/dmlcz/105746
ON SINGLEVALUEDNESS AND (STRONG) UPPER SEMICONTINUITY OF MAXIMAL MONOTONE MAPPINGS

Marian FABIAN, Praha

Abstract: Under suitable assumptions on the geometry of a dual $X^*$ of a real Banach space $X$ it is shown that a maximal monotone multivalued mapping $T$ from $X$ to $X^*$ with $\text{int} \ D(T) \neq \emptyset$ is singlevalued and upper semicontinuous on a dense residual subset of $\text{int} \ D(T)$.

Key words: Banach space, demiclosed multivalued mapping, singlevaluedness, upper semicontinuity, differentiability.

AMS: Primary: 47H05 Ref. Z.: 7.978.425
Secondary 54C05, 46B05 3.966, 7.972.25

Introduction. Let $X$ be a real Banach space with a topological dual $X^*$, $T: X \rightarrow 2^{X^*}$ a maximal monotone multivalued mapping whose domain has nonempty interior, i.e., $\text{int} \ D(T) \neq \emptyset$. Two theorems are the main result of this paper, which we can formulate roughly as follows:

Theorem A (on singlevaluedness of $T$). If the dual $X^*$ is strictly convex, then the set

$MV(T) = \{ x \in D(T) | T(x) \text{ is not a singleton} \}$

is of the first (Baire's) category in $X$.

Theorem B (on (strong) upper semicontinuity of $T$). If the dual $X^*$ is strictly convex and has the property (R)
(i.e., if $\{w_n\}_n \subset X^*$ converges weakly* to $w \in X^*$ and $\|w_n\| \to \|w\|$, then $w_n \to w$), then there exists a set $C \subset \text{int } D(T)$ dense residual in $\text{int } D(T)$ such that for every $x \in C$ the set $T(x)$ is a singleton and $T$ is upper semicontinuous at $x$, i.e., for $u \in D(T)$ sufficiently close to $x$, the set $T(u)$ lies in an arbitrary small given (norm) neighbourhood of $T(x)$.

See for details Theorems 2.1 - 2.3, Remarks 2.2 - 2.4 and the definition formulas (2.1) and (2.2).

Let us recall that the property of a mapping $T: X \to X^{**}$ to be maximal monotone is independent of which equivalent norm is taken in $X$. Hence, by using the renorming statement of Amir and Lindenstrauss [2], we obtain that the conclusion of Theorem A holds for any WCG $X$, especially, for $X$ reflexive or separable. It follows from the renorming statement of John and Zizler [7] that the conclusion of Theorem B is valid for such WCG $X$ which have a WCG dual $X^{**}$ (more generally, for those WCG $X$ which have an equivalent Fréchet differentiable norm, see [8]), especially, for $X$ reflexive or such $X$ whose dual $X^{**}$ is separable.

Using the simple fact that a subdifferential of a convex lower semicontinuous function is a monotone multivalued mapping, we get, from Theorems A and B, the well-known results of Asplund [3] concerning the Gâteaux and Fréchet differentiability of convex functions, see Remark 2.6.

The theorem on singlevaluedness of $T$ for $X$ separable has been proved by Zarantonello [21] in a geometrical way, later, topologically, by Kenderov [12] and Robert [16] and
more generally, for \( X \) with a strictly convex dual \( X^* \) by Kenderov [10]. Our Theorem 2.1 is a little improvement of Kenderov's result [10], where it is supposed \( D(T) = X \).

The theorem dealing with (strong) upper semicontinuity of \( T \) for \( X \) with a separable dual \( X^* \) has been proved by Robert [17].

The present paper was stimulated by the ideas of Kenderov [10], by means of which he derives the theorem on single-valuedness of \( T \). In doing so he uses the well-known deep fact that \( T \) is weakly* upper semicontinuous at each \( x \in \text{int} \ D(T) \). However, one can do with the demiclosedness of \( T \) only, which is a simple property of maximal monotone mappings.

In this paper, the ideas of Kenderov [10] are generalized to demiclosed multivalued mappings from a metric space \( P \) to a dual \( X^* \) (see Lemmas 1.1 - 1.3) and extended to the study of the (strong) continuity of such mappings (see Lemma 1.4), and so we get the topological means to prove Theorems 2.1 - 2.3.

The method proposed can be also used for the study of maximal accretive mappings (see, e.g.,[13] for definition).

The author would like to express his deepest gratitude to Josef Kolomy for advice and many helpful suggestions.

§ 0. Preliminaries. Let \( U, V \) be arbitrary sets. Then each nonempty subset \( T \) of \( U \times V \) is called a multivalued mapping from \( U \) to \( V \) and we write \( T: U \to 2^V \). The set \( T^{-1} = \{(v,u) \in V \times U \mid (u,v) \in T\} \) is called the inverse multivalued mapping to \( T \). Thus \( T^{-1}: V \to 2^U \). Obviously, \((T^{-1})^{-1} = T\).
For each \( u \in U \), we set
\[
T(u) = \{ v \in V \mid (u,v) \in T \}.
\]
If the set \( T(u) \) consists of one point only, we denote this point by the symbol \( T(u) \), too. The set
\[
D(T) = \{ u \in U \mid T(u) \neq \emptyset \}
\]
is called the domain of \( T \), the set \( R(T) = D(T^{-1}) \), the range of \( T \). It is introduced by many authors the graph \( G(T) \) of a multivalued mapping \( T \) by
\[
G(T) = \{ (u,v) \in U \times V \mid v \in T(u) \}.
\]
Obviously, \( G(T) \) coincides with \( T \). Therefore we shall not distinguish between a multivalued mapping and its graph.

A subset \( T \subseteq U \times V \) is called a singlevalued mapping, if the following implication holds:
\[
(u,v_1), (u,v_2) \in T \implies v_1 = v_2.
\]
In this case, we write \( T: U \to V \).

A subset \( T_1 \subseteq T \subseteq U \times V \) is called a selection of the multivalued mapping \( T \), if \( T_1 \) is singlevalued and \( D(T_1) = D(T) \).

Throughout the paper \( R \) will denote the set of real numbers endowed with the usual topology, \( X \) a real normed linear space, \( X^* \) its topological dual (the norm on \( X^* \) is dual to the norm on \( X \)), \( P \) a metric space. If \( A \) is a subset of \( P \), then \( \text{int} \ A \) will denote the topological interior of \( A \) and \( \text{cl} \ A \) the closure of \( A \). We recall that a subset \( A \subseteq P \) is called residual in \( P \) if the set \( P \setminus A \) is of the first (Baire's) category in \( P \). The arrows \( \to \), \( \to \) will denote the strong and weak* convergence, respectively.

A singlevalued mapping \( f: P \to R \cup \{ +\infty \} \) is called a function. The set

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dom f = \{ u \in D(f) \mid f(u) < + \infty \}

is called the effective domain of f.
A function f is said to be lower semicontinuous if
\[ \forall a \in \mathbb{R} \left( \text{the set} \left\{ u \in P \mid f(u) \leq a \right\} \text{is closed} \right) \]

Let T: P \rightarrow X^* be a singlevalued mapping from a metric space P to a dual X^* and let u \in D(T). T is said to be demicontinuous at u if
\[ \forall \text{sequence} \left( u_n \right) \subset D(T) \left[ u_n \rightarrow u \rightarrow T(u_n) \rightarrow T(u) \right], \]
Let T: P \rightarrow 2^{X^*} be a multivalued mapping from a metric space P to a dual X^*. T is said to be demiclosed if
\[ \forall u \in P \\forall w \in X^* \\forall \text{net} \left\{ (w_\alpha, w_\gamma) \mid \alpha \in \Lambda \right\} \subset T \]
\[ \left((u_\alpha \rightarrow u(\Lambda)), w_\alpha \rightarrow w(\Lambda), \sup \left\{ \| w_\alpha \| \mid \alpha \in \Lambda \right\} < + \infty \right) \rightarrow \]
\[ \rightarrow (u, w) \in T. \]

Let T: X \rightarrow 2^{X^*} be a multivalued mapping from a real normed linear space X to its dual X^*. T is said to be monotone if (for x \in X and x^* \in X^* the symbol \langle x^*, x \rangle denotes the value of the functional x^* at x)
\[ \forall (x, x^*) \in T \\forall (y, y^*) \in T \left[ \langle x^* - y^*, x - y \rangle \geq 0 \right], \]
and maximal monotone if T is not properly contained in any other monotone mapping.

§ 1. Lemma on continuity of demiclosed mappings.

Lemma 1.1. Let T: P \rightarrow 2^{X^*} be a demiclosed multivalued mapping from a metric space P to a dual X^* of a normed linear space X.
Then the function $f_T : P \rightarrow R \cup \{+\infty\}$ defined by

(1.1) $f_T(u) = \inf \{ \|w\| \mid w \in T(u) \}, \ u \in P$

is lower semicontinuous.

**Proof:** Let $a \in R$ be arbitrary. We have to show that the set

$A = \{ u \in P \mid f_T(u) \leq a \}$

is closed. Let $u \in \text{cl} A$ and let $\{ u_n \} \subseteq A$ be a sequence such that $u_n \rightarrow u$. For each $n = 1, 2, \ldots$, we find $w_n \in T(u_n)$ such that

$f_T(u_n) \leq \|w_n\| \leq f_T(u_n) + 1/n.$

Thus

(1.2) $\|w_n\| \leq a + 1/n, \ n = 1, 2, \ldots$

and so the sequence $\{w_n\}$ is bounded, hence $w^\ast$-praeom pact. Therefore there is $w \in X^\ast$ and a subnet $\{w_{n_\alpha} \mid \alpha \in \Lambda \}$ of the sequence $\{w_n\}$ such that

(1.3) $w_{n_\alpha} \rightarrow w(\Lambda).$

And since $u_{n_\alpha} \rightarrow u(\Lambda)$, too, and $T$ is demiclosed, $(u, w) \in T$. From the weak* lower semicontinuity ($w^\ast$.l.s.c., in abbreviation) of the norm on $X^\ast$, by using (1.2) and (1.3), we have

$\|w\| \leq \lim \inf_{\alpha \in \Lambda} \|w_{n_\alpha}\| \leq a.$

Thus $f_T(u) \leq \|w\| \leq a$, i.e., $u \in A$. The closedness of $A$ is proved, which completes the proof. Q.E.D.
We recall two well-known propositions.

**Proposition 1.1** ([5, 13.4]). If \( S: P \to Q \) is a single-valued mapping from a metric space \( P \) to a metric space \( Q \), then the set \( C(S) \) of all those points at which \( S \) is continuous, is \( G_\sigma \) in \( D(S) \), i.e., the set \( NC(S) = D(S) \setminus C(S) \) is \( F_\sigma \) in \( D(S) \).

**Proposition 1.2.** Let \( P \) be a metric space and \( f: P \to R \cup \{+\infty\} \) a lower semicontinuous function. Then the set \( C(f) \) of all those points at which \( f \) is continuous, is residual in \( \text{dom} \ f \), i.e., the set \( NC(f) = \text{dom} \ f \setminus C(f) \) is of the first (Baire's) category in \( \text{dom} \ f \).

**Proof:** See 14.7.6 and 14.5.2 in [5].

**Lemma 1.2.** Let \( T: P \to 2^{X^*} \) be a demiclosed multivalued mapping from a metric space \( P \) to a dual \( X^* \) of a normed linear space \( X \). Let the function \( f_T \) be defined by (1.1).

Then the set \( C(f_T) \) of all those points at which \( f_T \) is continuous, is residual \( G_\sigma \) in \( D(T) \).

**Proof:** It follows immediately from Lemma 1.1 and Propositions 1.1 and 1.2.

Let \( T: P \to 2^{X^*} \) be a multivalued mapping. A selection \( T_0 \) of \( T \) is said to be lower (with respect to the norm on \( X^* \)), if

\[
(1.4) \quad (u,w) \in T \iff \|T_0(u)\| \leq \|w\|.
\]

Obviously,

\[
(1.5) \quad \|T_0(u)\| = f_T(u) \text{ for } u \in D(T).
\]

We shall show that if \( T \) is demiclosed, then there exists
at least one lower selection of $T$. Let $u \in D(T)$ be arbitrary. Denote $c = \inf \{ \| w \| \mid w \in T(u) \}$ and set

$$K = \{ w \in T(u) \mid \| w \| \leq c + 1 \}.$$ 

Then $K$ is a nonempty bounded and $w^*$-closed subset of $X^*$, hence $w^*$-compact. So the norm on $X^*$, which is $w^*$.l.s.c., attains its minimum on $K$, i.e., there is a $w_o \in K \subset T(u)$ such that $\| w_o \| = c$.

For every singlevalued mapping $S: P \rightarrow X^*$, we introduce the sets

$$C^d(S) = \{ u \in D(S) \mid S \text{ is demicontinuous at } u \},$$

$$NC^d(S) = D(S) \setminus C^d(S).$$

**Lemma 1.3.** Let $T: P \rightarrow 2^{X^*}$ be a demiclosed multivalued mapping from a metric space $P$ to a dual $X^*$ of a normed linear space $X$, $f_T$ the function defined by (1.1). Let there exist a unique lower selection $T_o$ of $T$.

Then, if $f_T$ is continuous at $u \in D(T)$, $T_o$ is demicontinuous at $u$:

$$(1.6) \quad C(f_T) \subset C(T_o), \text{ i.e., } NC^d(T_o) \subset NC(f_T)$$

and hence, the set $C^d(T_o)$ is residual in $D(T)$.

**Proof:** Let $u \in C(f_T)$ be arbitrary. Let $\{ u_n \}$ be a sequence in $D(T)$ such that $u_n \rightarrow u$. Since (1.5) holds and $u \in C(f_T)$,

$$(1.7) \quad \| T_o(u_n) \| \rightarrow \| T_o(u) \|,$$

hence, the sequence $\{ T_o(u_n) \}$ is bounded. It implies that from any subsequence of $\{ T_o(u_n) \}$, we can extract a subnet converging weakly* to some $w \in X^*$. Then the demiclosedness
of \( T \) gives that \((u,w)\in T\), hence, by (1.4), \( \|w\| \leq \|T_0(u)\| \). But \( w^* \) l.s.c. of the norm on \( X^* \), and (1.7) implies \( \|w\| \leq \|T_0(u)\| \). Thus \( \|w\| = \|T_0(u)\| \). From here, and from the uniqueness of the lower selection of \( T \), we obtain \( w = T_0(u) \). It means that the whole sequence \( \{ T_0(u_n) \} \) is converging weakly* to \( T_0(u) \), so that \( u \in C^d(T_0) \), which proves (1.6). Finally, it follows from (1.6), by Lemma 1.2, that the set \( C^d(T_0) \) is residual in \( D(T) \). Q.E.D.

**Corollary 1.1.** Let \( S: P \rightarrow X^* \) be a demiclosed single-valued mapping from a metric space \( P \) to a dual \( X^* \) of a normed linear space \( X \).

Then the set \( C^d(S) \) of all those points at which \( S \) is demicontinuous, is residual in \( D(S) \).

**Lemma 1.4.** Let \( P \) be a metric space and \( X \) a normed linear space whose dual \( X^* \) has the property \((H)\). Let \( T: P \rightarrow 2^{X^*} \) be a demiclosed multivalued mapping and let there exist a unique lower selection \( T_0 \) of \( T \). Let \( f_T \) be the function defined by (1.1).

Then \( T_0 \) is continuous at \( u \in D(T) \) iff \( f_T \) is continuous at \( u \):

\[
C(T_0) = C(f_T) \text{, i.e., } NC(T_0) = NC(f_T)
\]

and hence, the set \( C(T_0) \) is residual \( Gc^r \) in \( D(T) \).

**Proof:** Since \( X^* \) has the property \((H)\), for every \( w \in X^* \) and for every sequence \( \{ w_n \} \subset X^* \), the following equivalence holds

\[
w_n \rightarrow w \iff (w_n \rightarrow w \text{ and } \|w_n\| \rightarrow \|w\|).
\]

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Let \( u \in D(T) \) and let \( \{ u_n \} \) be a sequence in \( D(T) \) such that \( u_n \xrightarrow{w} u \). If we set \( w = T_0(u) \) and \( w_n = T_0(u_n) \), \( n = 1, 2, \ldots \) in (1.9), we obtain

\[
T_0(u_n) \longrightarrow T_0(u) \iff (T_0(u_n) \longrightarrow T_0(u) \text{ and } \| T_0(u_n) \| \longrightarrow \| T_0(u) \|).
\]

Therefore (see (1.5)),

\[
C(T_0) = C(f_T) \cap C^d(T_0).
\]

But, by Lemma 1.3, we have \( C(f_T) \subseteq C^d(T_0) \), thus (1.8) holds. The rest of the conclusion of the Lemma follows from the identity (1.8) by Lemma 1.2. Q.E.D.

**Corollary 1.2.** Let \( P \) be a metric space, \( X \) a normed linear space whose dual \( X^* \) has the property (H). Let \( S: P \rightarrow X^* \) be a demiclosed singlevalued mapping.

Then the set \( C(S) \) of all those points at which \( S \) is continuous, is residual \( G_\sigma^* \) in \( D(S) \).

**Corollary 1.3.** Let \( S: P \rightarrow X \) be a demiclosed singlevalued mapping from a metric space \( P \) to a reflexive Banach space \( X \). Then the set \( C(S) \) of all those points at which \( S \) is continuous, is residual \( G_\sigma^* \) in \( D(S) \).

**Proof:** It follows immediately from the renorming statement of Troyanski [20] by Corollary 1.2, where we write \( X^* \) instead of \( X \).

It should be noted that, in the book of Alexiewicz [1, V.2.1.], there is a similar statement for \( X \) separable:

Let \( S: P \rightarrow X \) be a singlevalued mapping (with \( D(S) = P \)) from a complete metric space \( P \) to a separable normed
linear space $X$ such that

\[ u_n \to u \implies \langle x^*, S(u_n) \rangle \to \langle x^*, S(u) \rangle \text{ for every } x^* \in Z^*, \]

where $Z^*$ is such a subset of $X^*$ that for every $x \in X$,

\[ \| x \| = \sup \{ \langle x^*, x \rangle \mid x^* \in Z^*, \| x^* \| \leq 1 \}. \]

Then the set $NC(S)$ of all those points at which $S$ is not continuous, is of the first category in $P$.

If $S: Y \to X$ is a singlevalued linear closed (i.e., $y_n \to y$ and $S(y_n) \to x$ imply $y \in D(S)$ and $x = S(y)$) mapping from a normed linear space $Y$ to a reflexive Banach space $X$, with $D(S)$ of the second category in itself, we receive from Corollary 1.3 with help of Mazur's theorem that $S$ is continuous, which is a special case of Banach's closed graph theorem.

§ 2. Theorems on singlevaluedness and (strong) upper semicontinuity of maximal monotone mappings

We start by the following simple lemma:

**Lemma 2.1.** A maximal monotone multivalued mapping $T: X \to 2^{X^*}$ from a normed linear space $X$ to its dual $X^*$ is demiclosed and has at least one lower selection.

If, in addition, $X^*$ is strictly convex, there is a unique lower selection $T_0$ of $T$.

**Proof:** Let $\{ (x_{\alpha}, w_{\alpha}) \mid \alpha \in \Lambda \}$ be a set in $T$ such that

\[ x_{\alpha} \to x(\Lambda), w_{\alpha} \to w(\Lambda), \sup \{ \| w_{\alpha} \| \mid \alpha \in \Lambda \} < +\infty. \]
Let \((y, y^*) \in T\) be arbitrary. From the monotonicity of \(T\), we have

\[
\langle w_\alpha - y^*, x_\alpha - y \rangle \geq 0 \text{ for all } \alpha \in \Lambda,
\]

and passing to a limit, we get \(\langle w - y^*, x - y \rangle \geq 0\). Since \((y, y^*) \in T\) was arbitrary, the maximal monotonicity of \(T\) gives \((x, w) \in T\). Thus the demiclosedness of \(T\) is proved and therefore \(T\) has at least one lower selection.

Further, let \(X^*\) be strictly convex. Suppose that for some \(x \in D(T)\), there are \(w, z \in T(x)\) such that \(\|w\| = \|z\| = c = \inf \{\|x^*\| \mid x^* \in T(x)\}\). Then the convexity of \(T(x)\) (see, e.g., [4]) gives \((w + z)/2 \in T(x)\), hence \(\|(w + z)/2\| \geq c\). But, on the other hand, \(\|(w + z)/2\| \leq c/2 + c/2 = c\). Thus the strict convexity of \(X^*\) yields \(w = z\). Hence, two different lower selections of \(T\) cannot exist. Q.E.D.

Let \(M\) be a nonempty subset of a normed linear space \(X\). Following Kato [9], we introduce the set

\[
\text{dint } M = \{x \in M \mid c1 \phi_x(M) = X\},
\]

where

\[
\phi_x(M) = \{u \in X \mid \exists \{t_n\} \subset \mathbb{R}, t_n \to 0, t_n \downarrow 0, \in \{x + t_n u \mid c M\} \}.
\]

It should be noted that \(\text{int } M\) and the algebraic interior of \(M\) even are included in \(\text{dint } M\).

**Example 2.1.** Let \(H\) be a separable Hilbert space, \(\{e_i\}\) a total orthonormal system in \(H\). We set

\[
M = \{x = \sum_{i \in I} t_i e_i \mid \{t_i\} \subset \mathbb{R}, \sum_{i \in I} |t_i|^2 \leq 1\}.
\]
It is easy to show that the set $M$ is convex closed (hence, of the second category in itself) having empty algebraic interior, but $\text{dint } M \neq \emptyset$, even $M = \text{cl} (\text{dint } M)$. 

**Lemma 2.2.** Let $T : X \rightarrow 2^{X^*}$ be a monotone multivalued mapping from a normed linear space $X$ to its dual $X^*$ and let $T_1$ be an arbitrary selection of $T$. Denote

$$\text{SV}(T) = \{ x \in \text{D}(T) \mid T(x) \text{ is a singleton} \} ,$$

$$\text{MV}(T) = \text{D}(T) \setminus \text{SV}(T).$$

Then, if $T_1$ is demicontinuous at $x \in \text{dint } \text{D}(T)$, the set $T(x)$ is a singleton:

$$\tag{2.4} \text{C}^d(T_1) \cap \text{dint } \text{D}(T) \subseteq \text{SV}(T), \text{ i.e.},$$

$$\text{MV}(T) \cap \text{dint } \text{D}(T) \subseteq \text{NC}^d(T_1).$$

**Proof:** Let $x \in \text{C}^d(T_1) \cap \text{dint } \text{D}(T)$. Let $w$ be an arbitrary element of the set $T(x)$. For every $u \in F_x(\text{D}(T))$ and the corresponding sequence $\{ t_n \mid t_n > 0, t_n \downarrow 0 \}$ (see (2.1) and (2.2)), from the monotonicity of $T$, we have

$$\langle T_1(x + t_n u) - w, (x + t_n u) - x \rangle \geq 0, \quad n = 1, 2, \ldots ,$$

and cancelling it by $t_n \overset{0}$,

$$\langle T_1(x + t_n u) - w, u \rangle \geq 0, \quad n = 1, 2, \ldots .$$

Using the demicontinuity (even the hemi-continuity only) of $T$, we then obtain that

$$\langle T_1(x) - w, u \rangle \geq 0.$$

Since this inequality holds for each $u \in F_x(\text{D}(T))$, and $F_x(\text{D}(T))$ is a dense subset in $X$, it must be $T_1(x) = w$. But
w was arbitrary element of the set $T(x)$, hence $T(x)$ is a singleton, i.e., $x \in SV(T)$. Thus the lemma is proved. Q.E.D.

**Remark 2.1.** If $T: X \rightarrow 2^{X^*}$ is a maximal monotone multivalued mapping from a Banach space $X$ to $X^*$, with int $D(T) \neq \emptyset$, (2.4) can be strengthened. The result of Rockafellar [18] says that $SV(T) \subset int D(T)$ and that $T$ is locally bounded at any point of int $D(T)$. From this and from (2.4), we can derive the following identity

$$C^d(T_1) \cap int D(T) = SV(T).$$

**Theorem 2.1.** Let $X$ be a Banach space with a strictly convex dual $X^*$ and $T: X \rightarrow 2^{X^*}$ a maximal monotone multivalued mapping.

Then the set

$$MV(T) \cap int D(T) = \{x \in int D(T) \mid T(x) \text{ is not a singleton}\}$$

is of the first category in $D(T)$.

If, moreover, int $D(T) \neq \emptyset$, then the set

$$SV(T) \cap int D(T) = \{x \in int D(T) \mid T(x) \text{ is a singleton}\}$$

is dense residual in int $D(T)$.

**Proof:** The first assertion follows immediately from Lemmas 2.2 and 1.3.

Further, let int $D(T) \neq \emptyset$. Since the obvious inclusion int $D(T) \subset int D(T)$ holds, the set $MV(T) \cap int D(T)$ is of the first category in $D(T)$, hence also in $X$ and in the open non-empty set int $D(T)$. Therefore the set

$$SV(T) \cap int D(T) = int D(T) \setminus (MV(T) \cap int D(T))$$

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is residual in \( \mathrm{int} \ D(T) \) and, by Baire's category theorem, is dense in \( \mathrm{int} \ D(T) \). Q.E.D.

**Remark 2.2.** Since \( \mbox{SV}(T) \subset \mathrm{int} \ D(T) \) (see [18]), we can write \( \mbox{SV}(T) \) instead of \( \mbox{SV}(T) \cap \mathrm{int} \ D(T) \) in Theorem 2.1.

**Theorem 2.2.** Let \( X \) be a Banach space with a dual \( X^* \) which is strictly convex and has the property (H). Let \( T: X \to 2^{X^*} \) be a maximal monotone multivalued mapping. Then:

(i) There exists a unique lower selection \( T_0 \) of \( T \).

(ii) For each \( x \in \mathrm{dint} \ D(T) \) at which \( T_0 \) is continuous, \( T(x) \) is a singleton.

(iii) The set \( C(T_0) \) of all those points at which \( T_0 \) is continuous, is residual \( G_\delta \) in \( D(T) \), i.e., the set \( N(C(T_0)) = D(T) \setminus C(T_0) \) is of the first category \( F_\sigma \) in \( D(T) \).

(iv) If, in addition, \( \mathrm{int} \ D(T) \neq \emptyset \), the set \( C(T_0) \cap \mathrm{int} \ D(T) \) is dense residual \( G_\delta \) in \( \mathrm{int} \ D(T) \).

**Proof:** (i) is contained in Lemma 2.1. (ii) follows from Lemma 2.2 and the obvious inclusion \( C(T_0) \subset C(T_0) \). (iii) is obtained by using (i) and Lemma 1.4. (iv) follows from (iii) and Baire's category theorem. Q.E.D.

**Example 2.2.** Let \( H \) be a separable Hilbert space, \( \{ e_i \} \) a total orthonormal system in \( H \) and \( M \subset H \) the set defined by (2.3). Define the function \( \varphi: H \to \mathbb{R} \cup \{ +\infty \} \) as follows

\[
\varphi(x) = 0, \text{ if } x \in M, \quad \varphi(x) = +\infty, \text{ if } x \notin M.
\]

Obviously, \( \varphi \) is a convex lower semicontinuous function. By [19], the subdifferential \( \partial \varphi \) of \( \varphi \) is a maximal monotone multivalued mapping from \( H \) to \( H \), with \( D(\partial \varphi) = M \). Hen-
ce, according to Example 2.1, $\text{int } D(\partial g) = \emptyset$, but $\text{dint } D(\partial g) \neq \emptyset$, and $\text{cl } (\text{dint } D(\partial g)) = D(\partial g)$ is of the second category in itself. It justifies the extension of our reasoning from the class of maximal monotone mappings $T$, with $\text{int } D(T) \neq \emptyset$, to that, with $\text{dint } D(T) \neq \emptyset$.

If $\text{int } D(T) \neq \emptyset$, then for the points $x \in C(T_0) \cap \text{int } D(T)$, we shall derive a little more still, namely, that at such points $x$, the mapping $T$ is (strongly) upper semicontinuous. We shall use the following lemma.

**Lemma 2.3.** Let $T: X \rightarrow 2^{X^*}$ be a monotone multivalued mapping from a normed linear space $X$ to its dual $X^*$ such that $\text{int } D(T) \neq \emptyset$ and let $T_1, T_2$ be two arbitrary selections of $T$. Denote by $C(T_1), C(T_2)$ the sets of all those points at which $T_1, T_2$ are continuous, respectively. Then

$$C(T_1) \cap \text{int } D(T) = C(T_2) \cap \text{int } D(T).$$

**Proof:** In view of the symmetry of the conclusion, it suffices to prove the inclusion $\subseteq$ in (2.5). Let $x \in C(T_1) \cap \text{int } D(T)$ be arbitrary. Recall that, by Lemma 2.2, $T_1(x) = T_2(x) = T(x)$. Let $\{x_n\} \subset \text{int } D(T)$ be a sequence such that $x_n \rightarrow x$. Since $x \in \text{int } D(T)$, we can suppose that $\{x_n\} \subset \text{cl } \text{int } D(T)$. For each $n = 1, 2, \ldots$, we find $v_n \in X$ so that

$$\|v_n\| \leq 1 \text{ and } \|T_2(x_n) - T(x)\| - 1/n \leq \\
\leq \langle T_2(x_n) - T(x), v_n \rangle.$$  

Further, for every $n = 1, 2, \ldots$, we choose $t_n \in (0, 1/n)$ so that $x_n + t_n v_n \in \text{D}(T)$.

The monotonicity of $T$ gives
\[ \langle T_1(x_n + t_n v_n) - T_2(x_n), (x_n + t_n v_n) - x_n \rangle \geq 0, \]
hence
\[ \langle T_2(x_n), v_n \rangle \leq \langle T_1(x_n + t_n v_n), v_n \rangle, \]
which together with (2.6) yields
\[ \| T_2(x_n) - T(x) \| - 1/n \leq \langle T_1(x_n + t_n v_n) - T(x), v_n \rangle \leq\]
\[ \leq \| T_1(x_n + t_n v_n) - T(x) \|. \]
But \( x_n + t_n v_n \to x \) and \( x \in \text{C}(T_1) \). Therefore the last inequality gives that \( \| T_2(x_n) - T(x) \| \to 0 \), i.e., \( x \in \text{C}(T_2) \).

Q.E.D.

**Theorem 2.3.** Let \( X \) be a Banach space with a dual \( X^* \) which is strictly convex and has the property (H). Let \( T: X \to 2^{X^*} \) be a maximal monotone multivalued mapping with \( \text{int D}(T) \neq \emptyset \).

Then the set of all those \( x \in \text{int D}(T) \) for which the set \( T(x) \) is a singleton and \( T \) is upper semicontinuous at \( x \), i.e., given \( \varepsilon > 0 \), there exists \( \delta^* > 0 \) such that for each \( u \in \text{D}(T) \) fulfilling \( \| x - u \| < \delta^* \), the set \( T(u) \) is included in the \( \varepsilon \)-neighbourhood of \( T(x) \), is dense residual \( \text{G}_{\delta^*} \) in \( \text{int D}(T) \).

**Proof:** We set
\[ C = \text{int D}(T) \cap \text{C}(T_1), \]
where \( T_1 \) is an arbitrary selection of \( T \). (Thanks to Lemma 2.3, the set \( C \) does not depend on the choice of \( T_1 \).) By Theorem 2.2 (iv), \( C \) is dense residual \( \text{G}_{\delta^*} \) in \( \text{int D}(T) \). We shall show that \( C \) is that set of Theorem 2.3. Let \( x \in \text{int D}(T) \) be
such that $T(x)$ is a singleton and $T$ is upper semicontinuous at $x$. Then we easily get $x \in C(T)$, hence $x \in C$. Conversely, let $x \in C$ be arbitrary. By Lemma 2.2, the set $T(x)$ is a singleton. We shall be proving that $T$ is upper semicontinuous at $x$. Let us suppose the contrary. Then there exists an $\varepsilon > 0$ and a sequence $\{(u_n, w_n)\} \subset T$ such that $u_n \rightarrow x$ and

$$\|w_n - T(x)\| \leq \varepsilon, \quad n = 1, 2, \ldots.$$  \hspace{1cm} (2.7)

We define the selection $T_2$ of $T$ as follows:

$$T_2(u_n) = w_n, \quad n = 1, 2, \ldots,$$

$$T_2(u) = \text{an arbitrary element of } T(u), \text{ for } u \notin \{u_n\}.$$

But since, by Lemma 2.3, $x \in C \subset C(T_2)$,

$$w_n = T_2(u_n) \rightarrow T_2(x) = T(x),$$

which is in contradiction with (2.7). It means $T$ is upper semicontinuous at $x$. Q.E.D.

Remark 2.3. The second part of Theorem 2.1, and Theorem 2.3 are valid for arbitrary monotone multivalued mapping $T : X \rightarrow 2^{X^*}$, with $\text{int } D(T) \neq \emptyset$.

Remark 2.4. Let $T : X \rightarrow 2^{X^*}$ be a maximal monotone multivalued mapping from a Banach space $X$ to its dual $X^*$ such that $\text{int } D(T) \neq \emptyset$. Then, by Rockafellar's result [18],

$D(T) \subset \text{cl } (\text{int } D(T))$, and hence, the set $D(T) \setminus \text{int } D(T)$ is nowhere dense in $D(T)$. Therefore the text "in $\text{int } D(T)$" in Theorems 2.1 - 2.3 can be replaced by "in $D(T)$" (provided that $T$ is maximal monotone).
Remark 2.5. A somewhat different method for obtaining the results above, in the special case when $X$ is reflexive, is given in [6].

Remark 2.6. Let $f: X \rightarrow \mathbb{R} \cup \{+ \infty\}$ be a convex lower semicontinuous function, with $D(f) = X$ and $\text{int} \ (\text{dom} \ f) \neq \emptyset$. Then, it can be easily seen that the subdifferential $\partial f$ of $f$ is a monotone multivalued mapping. Using [14], we immediately derive from Theorem 2.1 and Remark 2.3 that if $X^*$ is strictly convex, then the set of those points at which $f$ is Gâteaux differentiable, is dense residual in $\text{int} \ (\text{dom} \ f)$, which is included in Theorem 2 in [3]. It follows from Theorem 2.3 and Remark 2.3 by means of Proposition (ii) in [17] that if $X^*$ is strictly convex and has the property (H), then the set of those points at which $f$ is Fréchet differentiable, is dense residual $G^r$ in $\text{int} \ (\text{dom} \ f)$. This result is a little stronger than Theorem 1 in [3], where it is required for $X^*$ to be locally uniformly convex. However, our statement is included in [15].

Added in proof. After this paper had been prepared for publication, the author received the preprint by P. Kenderov and R. Robert: Nouveaux résultats génériques sur les opérateurs monotones dans les espaces de Banach, which will appear in C.R. Acad. Sci. Paris. Here it is independently shown that the conclusion of Theorem B is valid, if $X^*$ has the property (H), where nets are taken instead of sequences, without the assumption of strict convexity of $X^*$.

From the sketch of the proofs in the quoted work, it is obvious that our methods of the proofs are rather different.
References


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