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TERNARY RINGS ASSOCIATED TO TRANSLATION PLANE

Josef KLOUDA, Praha

Abstract: It is well known that an affine plane is a translation plane if and only if there exists a quasifield coordinatizing it. Simple condition for planary ternary ring with zero coordinatizing a translation plane is deduced by Klucky and Marková in [4]. We shall define a J-ternary ring or JTR to be a PTR that exists S such that

\[ T(a,0,c) = T(a,b,c) \implies T(a,0,y) = T(a,b,y) \quad \forall y \in S \]

\[ T(0,a,c) = T(b,a,c) \implies T(0,a,y) = T(b,a,y) \quad \forall y \in S. \]

Structurally, the JTR lie between the PTR and ITR. The purpose of this note is to deduce a necessary and sufficient condition that a given JTR coordinatizes a translation plane. This generalizes the main results of [4] and [5].

Key words: Planar ternary ring, translation plane, intermediate ternary ring, generalized Cartesian group.

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A coordinatization of a projective plane: We shall give a coordinatization to a projective plane of order n. Let S be any set of cardinality n. Let \( \infty \) be any element which is not in S and let 0 \in S. We pick one point L and one line \( \ell \) joining through L in the plane. For any \( M \not\in \ell \) denote by \( \tilde{M} \) the line containing \( M \). Let \( m \mapsto (m) \) be a bijection of \( S \cup \{\infty\} \) onto \( \ell \) such that \( [\infty] = \ell \). Let \( x \mapsto [x] \) be a bijection of \( S \cup \{\infty\} \) onto \( \tilde{\ell} \) such that \( [\infty] = \tilde{\ell} \). Let \( y \mapsto (0,y) \) be a bijection of S onto \( \{01\} \times L \). We denote by \( A \cup B \) (amb) the line join-
ing two distinct points A, B (the common point of two distinct lines). Let \( \alpha_1, \alpha_2 : S \rightarrow S \) be two mappings. Then to every point \( P \) off \( \ell \) we assign coordinates \((x, y)\) if and only if \( P = \{x\} \cap ((\alpha_1(x)) \cup (0, y)) \). We shall now dualize the above construction in the following sense. Let \( c \mapsto [0, c] \) be a bijection of \( S \) onto \((0) \setminus \{1\}\). Then to every line \( p \) off \( \ell \) we assign coordinates \([m, c]\) if and only if \( p = (m) \cup ([\alpha_2(m)] \cap [0, c])\).

**Planar ternary rings:**

**Definition 1:** Let \( S \) be a set containing two different elements at least and let ternary operation \( T \) be given on it. An ordered pair \((S, T)\) will be called a planar ternary ring or PTR if it holds:

1. \( \forall a, b, c \in S \exists ! x \in S \quad T(a, b, x) = c \)
2. \( \forall a, b, c, d \in S; x \in S \quad T(x, a, b) = T(x, c, d) \)
3. \( \forall a, b, c, d \in S; a \neq c \exists (x, y) \in S^2 \quad T(a, x, y) = b, T(c, x, y) = d \)

An intermediate ternary ring on \( ITR \) (see [5], p. 1187) is a PTR \((S, T)\) such that \((I_1)\) and \((I_2)\) holds.

\( (I_1) \quad T(m, a, y) = T(m, b, y) = c, \quad a \not= b \implies T(m, x, y) = c \) \( \forall x \in S \)

\( (I_2) \quad T(a, x, y) = T(b, x, y) = c, \quad a \not= b \implies T(m, x, y) = c \) \( \forall x \in S \)

A \( J \)-ternary ring or JTR is a PTR \((S, T)\) such that there exists \( 0 \in S \) where

\( (J_1) \quad T(m, 0, a) = T(m, x, a) \implies T(m, 0, y) = T(m, x, y) \) \( \forall y \in S \)

\( (J_2) \quad T(0, x, a) = T(m, x, a) \implies T(0, x, y) = T(m, x, y) \) \( \forall y \in S \)
Let \((S, T)\) be a PTR. Then \((S, T)\) defines a projective plane \(\pi(S, T)\) as follows.

**Points:** \((x, y), (m), (\infty)\); \(m, x, y \in S\), \(\infty\) not in \(S\)

**Lines:**

\[
\begin{align*}
[m, c] &= \{ (x, y) \mid x, y \in S, \ T(m, x, y) = c \} \\
[x] &= \{ (x, y) \mid y \in S \} \\
[\infty] &= \{ (\infty) \} \cup \{ (m) \mid m \in S \}
\end{align*}
\]

In [2], [3](p. 114-115), [5](p. 1186) there was shown that \(\pi(S, T)\) is a projective plane. Thus a solution in (3) is unique.

**Proposition 1:** Let \(\pi\) be a projective plane. Then there exists a JTR \((S, T)\) such that \(\pi(S, T)\) is isomorphic to \(\pi\).

**Proof:** Let the projective plane \(\pi\) be coordinatized as above by elements from a set \(S\). Define a ternary operation by

\[T(m, x, y) = c\text{ if and only if } (x, y) \text{ is on } [m, c].\]

Then it is obvious that the \((S, T)\) is a JTR. One has only to check (1), (2), (3), \((J_1), (J_2)\) in turn.

**Remark:** Let \((S, T)\) be a JTR. Then there are mappings \(\alpha_1, \alpha_2 : S \rightarrow S\) such that

\[\forall x, y \in S \quad T(\alpha_1(x), 0, y) = T(\alpha_2(x), x, y)\]

and such that for every point \((x, y)\) and every line \([m, c]\) in \(\pi(S, T)\) is \((x, y) = [x] \cap ([\alpha_1(x)] \cup [0, y])\)

\([m, c] = [m] \cup ([\alpha_2(m)] \cap [0, c])\)

**Proposition 2:** Let \((S, T)\) be an ITR. Then \((S, T)\) is a JTR.

**Proof:** The proposition is a direct consequence of Theorem 6 in [5], p. 1188.

**Vertically transitive planes:** \((S, t)\) is said to be the dual ternary system of PTR \((S, T)\) if \(c_t = T(m, x, t(x, m, c))\)
\( \forall x, c, x \in \mathcal{S} \) or equivalently \( y = t(x, m, T(m, x, y)) \)

\( \forall m, x, y \in \mathcal{S} \).

**Proposition 3:** The dual of a JTR is a JTR.

**Proof:** The proof is straightforward.

In the following we shall denote by \( j^1_a \) the solution of the equation \( t(x, 0, 0) = t(x, a, a) \) for each \( a \in \mathcal{S} \setminus \{0\} \) and by \( j^2_a \) the solution of the equation \( T(x, 0, 0) = T(x, a, a) \) for each \( a \in \mathcal{S} \setminus \{0\} \); additionally we define \( j^1_0 = j^2_0 = 0 \). Thus for each \( a \in \mathcal{S} \), \( t(j^1_a, 0, 0) = t(j^1_a, a, a) \) and \( T(j^2_a, 0, 0) = T(j^2_a, a, a) \).

Now let us introduce in \( \mathcal{S} \) two binary operations \( +_1, +_2 \) by virtue of

\[
\begin{align*}
+_{1} : & \quad a +_1 b = T(a, j^1_a, t(j^1_a, 0, b)) \\
+_{2} : & \quad a +_2 b = t(a, j^2_a, T(j^2_a, 0, b)) \quad \forall a, b \in \mathcal{S}
\end{align*}
\]

**Remark:** It can be easily verified that

\( \forall a \in \mathcal{S} \)

\
\begin{align*}
(4) & \quad +_1 0 = 0, +_2 0 = 0 \\
(5) & \quad \forall a, b \in \mathcal{S} \exists! x \in \mathcal{S} \quad a +_1 x = b \\
& \quad \forall a, b \in \mathcal{S} \exists! y \in \mathcal{S} \quad a +_2 y = b
\end{align*}

**Definition 2:** Let \( (\mathcal{S}, T) \) be a PTR. The projective plane \( \pi(\mathcal{S}, T) \) is said to be a vertically transitive plane (by [4], p. 620) if for each \( x, y, z \in \mathcal{S} \) there exists a translation \( \tau \) of the affine plane \( (\mathcal{S}^2, \{[m, c] \mid m, c \in \mathcal{S}\} \cup \{[x] \mid x \in \mathcal{S}\}) \) such that \( (x, y) \tau = (x, z) \).

Let \( (\mathcal{S}, T) \) be a JTR and \( (\mathcal{S}, t) \) its dual. By (1)

\( \phi_1 : y \mapsto T(0, 0, y) \), \( \phi_2 : c \mapsto t(0, 0, c) \) are bijective mappings and \( \phi_1 \phi_2 = \phi_2 \phi_1 = \text{id} \).

**Proposition 4:** Let \( (\mathcal{S}, T) \) be a JTR. Then the projective plane \( \pi(\mathcal{S}, T) \) is a vertically transitive plane if and only if
Proof. I. Suppose first that \((S,T)\) (6) holds. We shall see that \((S,+_2)\) is a loop. By (4), (5) it is sufficient to show that \(\forall u, c \in S \exists v \in S \; v +_2 c = u\).

Let \(a +_2 c = b +_2 c\) and let \(m, x \in S\) such that \(x \neq 0\),

\[
T(m, 0, a) = T(m, x, b).
\]

Then

\[
T(m, 0, a +_2 c) = T(m, 0, a) + (0 +_2 c) + (0 +_2 c) = T(m, x, b +_2 c)
\]

by (J1)

\[
T(m, 0, a) = T(m, x, a) = T(m, x, b)
\]

hence \(a = b\). Now let \(u \in S\).

Choose \(m, x, y \in S\) such that \(x \neq 0\),

\[
T(m, 0, u) = T(m, x, y +_2 c)
\]

and denote

\((0, v) : = [m, T(m, x, y)] \cap [0, 1].\)

Then there is \(T(m, 0, v) = T(m, x, y), T(m, 0, u) = T(m, x, y +_2 c) = T(m, x, y) + (0 +_2 c) = T(m, 0, v) + (0 +_2 c) = T(m, 0, v +_2 c)\) from here \(v +_2 c = u\).

Thus, the map \(\varphi_c : S^2 \to S^2\) defined by

\[
(x, y) \varphi_c := (x, y +_2 c)
\]

is a translation. Since \((0, 0) \varphi_c = (0, c)\), the \(\pi(S, T)\) is a vertically transitive plane.

II. Let \(\pi(S, T)\) be a vertically transitive plane. Then for each \(a \in S\) there is a translation \(\varphi_a\) mapping \((0, 0)\) into \((0, a)\). Then \((y, y) \varphi_a = (y, y +_2 a)\) for each \(y \in S\) hence \((0, 0) \varphi_a = (0, y +_2 a)\) for each \(y \in S\) and \((x, y) \varphi_a = (x, y +_2 a)\) for each \(x, y \in S\). It is obvious that \([0, 0] \varphi_a = [0, (0 +_2 a) \varphi_a]\) this implies \([m, c] \varphi_a = [m, c +_1 (0 +_2 a) \varphi_a]\). Hence, \((x, y) \in [m, T(m, x, y)]\) for each \(m, x, y \in S\) from here \((x, y) \varphi_a \in [m, T(m, x, y)] \varphi_a\) then \((x, y +_2 a) \in [m, T(m, x, y) +_1 (0 +_2 a) \varphi_a]\) consequently \(T(m, x, y +_2 a) = T(m, x, y) +_1 (0 +_2 a) \varphi_a\) for each \(m, x, y, a \in S\).
Corollary 4.1: Let \((S,T)\) be a JTR and let \(\pi(S,T)\) be a vertically transitive plane. Then \((S,+_1)\), \((S,+_2)\) are groups and \((S,+_1)\) is isomorphic to \((S,+_2)\).

Proof: Consider translations \(\varphi: (0,0) \mapsto (0,a), \sigma: (0,0) \mapsto (0,b), \tau: (0,0) \mapsto (0,c).\) Then
\[
(0,(a + 2 b) +_2 c) = (0,0)(\varphi \sigma) = (0,0)(\tau \sigma) = (0,a +_2 (b + 2 c)).
\]

The second result follows from \((6)\). In particular, for every \(a,b \in S\) \((a +_2 b)\) \((a +_2 b)\) \(\vdash_1 = T(0,0,a +_2 b) = T(0,0,a) +_1 (0^{\varphi} +_2 b)\) \(= a^{\varphi} +_1 (0^{\varphi} +_2 b)\) \(\vdash_1 .\)

Since for each \(y,a,b \in S\)
\[
y +_2 (a +_2 b) = (y +_2 a) +_2 b,
\]
we have
\[
y^{\varphi} +_1 (0^{\varphi} +_2 (a +_1 b))^{\varphi} = (y^{\varphi} +_2 (a +_2 b))^{\varphi} =
\]
\[
(y +_2 a)^{\varphi} +_1 (0^{\varphi} +_2 b)^{\varphi} = (y^{\varphi} +_1 (0^{\varphi} +_2 a)^{\varphi}) +_1
\]
\[
+_1 (0^{\varphi} +_2 b)^{\varphi}.
\]
Setting \(y = 0^{\varphi} a\), we have
\[
(0^{\varphi} +_2 (a +_1 b))^{\varphi} = (0^{\varphi} +_2 a)^{\varphi} +_1 (0^{\varphi} +_2 b)^{\varphi}.
\]

Remark: The group of all translations of a vertically transitive plane \(\pi(S,T)\) is Abelian if and only if \((S,+_1)\) is commutative.

Now let us introduce two binary operations \(_{1}, \_2\) by virtue of

\[
T(m,x,0) = m \_1 x \quad \forall m,x \in S
\]
\[
t(x,m,0) = x \_2 m \quad \forall m,x \in S
\]

Corollary 4.2: Let \((S,T)\) be a JTR and let \(\pi(S,T)\) be a vertically transitive plane. Then
\[
(7) \forall m,x,y \in S \quad T(m,x,y) = m \_1 x +_1 (0^{\varphi} +_2 y)^{\varphi}
\]
\[
t(x,m,y) = x \_2 m +_2 (0^{\varphi} +_1 y)^{\varphi}
\]
Proof: Let as set \( y = 0 \) in (6). Then
\[
T(m,x,c) = m \cdot 1^x + 1 (0^2 + 2 c)^1
\]
for each \( m,x,c \in S \).

Proposition 5: Let \((S,T)\) be a JTR. The projective plane \( \pi(S,T) \) is a vertically transitive plane if and only if
(8) \((S,+_1), (S,+_2)\) are groups
(9) there exists an isomorphism \( \varphi : (S,+_2) \rightarrow (S,+_1) \) such that
\[
\forall m,x,y \in S \quad T(m,x,y) = m \cdot 1^x + 1 y^\varphi.
\]

Proof: I. Let (8),(9) hold for \((S,T)\). Then for each \( m,x,y,c \in S \)
\[
T(m,x,y +_2 c) = m \cdot 1^x +_1 (y +_2 c)^\varphi = m \cdot 1^x +_1 (y^\varphi +_1 c^\varphi) = (m \cdot 1^x +_1 y^\varphi) +_1 c^\varphi = T(m,x,y) +_1 c^\varphi.
\]
Setting \( m=x=0, y=0^2 \), we have \((0^2 + 2 c)^P = 0 +_1 c^\varphi\)
thus \( c^\varphi = (0^2 + 2 c)^P \) for each \( c \in S \) therefore
\[
T(m,x,y +_2 c) = T(m,x,y) +_1 (0^2 +_2 c)^P
\]
for each \( m,x,y,c \in S \).

II. The second part follows immediately from Corollary 4.1 and Corollary 4.2.

Corollary 5.1: Let \((S,T)\) be a JTR such that
\[
T(0,0,y) = y \quad \text{for each} \quad y \in S.
\]
Then the projective plane \( \pi(S,T) \) is a vertically transitive plane if and only if
(i) \((S,+_1)\) is a group
(ii) \( \forall m,x,y \in S \quad T(m,x,y) = m \cdot 1^x +_1 y \)

Proof: I. \( \forall m,x,y,c \in S \)
\[
T(m,x,y +_1 c) = m \cdot 1^x +_1 (y +_1 c)^\varphi = (m \cdot 1^x +_1 y)^\varphi +_1 c = T(m,x,y) +_1 c.
\]
Hence \( \pi(S,T) \) is a vertically transitive plane.

II. If \( \pi(S,T) \) is a vertically transitive plane, then
by Proposition 5 \((S,+_1)\) is a group and there exists an isomorphism \( \varphi : (S,+_2) \rightarrow (S,+_1) \) such that
\[
T(m,x,y) = m \cdot 1^x +_1 y^\varphi
\]
for each \( m,x,y \in S \). This yields then
\[
y = T(0,0,y) = 0 +_1 y^\varphi = y^\varphi
\]
for each \( y \in S \) hence
\[
T(m,x,y) = m \cdot 1^x +_1 y \quad \text{for each} \quad m,x,y \in S.
\]

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Corollary 5.2: Let \((S,T)\) be a JTR and \((S,t)\) its dual. Let \(\pi(S,T)\) be a vertically transitive plane, then there exists an isomorphism \(\varphi:(S,+_2)\rightarrow(S,+_1)\) such that
\[
\forall m,x,y \in S \quad T(m,x,y) = m \cdot 1 x +_1 y \varphi,
\]
\[
t(x,m,y) = x \cdot 2 m +_2 y \varphi^{-1}, \quad m \cdot 1 x +_1 (x \cdot 2 m) \varphi = 0.
\]

Proof: Since it holds \(T(m,x,t(x,m,0)) = 0\) for each \(m,x \in S\), we have \(m \cdot 1 x +_1 (x \cdot 2 m) \varphi = 0\). Since it holds \(T(m,x,t(x,m,y)) = y\) for each \(m,x,y \in S\), we obtain \(m \cdot 1 x +_1 (t(x,m,y)) \varphi = y\) thus \((t(x,m,y)) \varphi = t(x,m,y) \varphi^{-1}\) \(m \cdot 1 x +_1 y = (x \cdot 2 m) \varphi +_1 y\) from what you say \(t(x,m,y) = x \cdot 2 m +_2 y \varphi^{-1}\).

Definition 3: Let \(S\) be a set, \(\cdot, +\) two binary operations on \(S\). \((S,+, \cdot)\) will be called a generalized Cartesian group (see [4], p. 620) if \(S\) has two distinct elements at least and if it holds:
\[
(10) \quad (S,+) \text{ is a group}
\]
\[
(11) \quad \forall a,b,c \in S; a \neq b \exists ! x \in S \quad ax + xb = c
\]
\[
(12) \quad \forall a,b,c \in S; a \neq b \exists ! x \in S \quad ax - bx = c
\]

Proposition 6: Let \(C = (S,+, \cdot)\) be a generalized Cartesian group and let \(\varphi: S \rightarrow S\) be a bijection such that \(0 \varphi = 0\). If we define \(T(C,\varphi)\), \(m,x,y) = m \cdot x + y \varphi\) for each \(m,x,y \in S\) then \((S,T(C,\varphi))\) is a JTR and \(\pi(S,T(C,\varphi))\) is a vertically transitive plane.

Proof: The proof is straightforward. One has only to check (1), (2), (3), (J_1), (J_2), (8), (9) in turn.

Proposition 5 and Proposition 6 now imply the next

Theorem 1: Let \((S,T)\) be a JTR. Then the projective plane \(\pi(S,T)\) is a vertically transitive plane if and only if
\[
(i) \quad C = (S,+,1, \cdot_1) \quad \text{is a generalized Cartesian group}
\]

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(ii) there exists a bijection \( \varphi: S \rightarrow S \) such that \( O^\varphi = 0, \)
\( T = T(C, \varphi) \).

Translation planes: First we give some general remarks. Let us investigate a projective plane \( \pi = (P, L) \). Let us distinguish a line \( L \). Then by an affine plane \( \pi(L) \) we shall as usual mean the restriction of \( \pi \) to the incidence structure \( (P \setminus L, \{ m \setminus (m \cap L) | m \in L \setminus \{L\} \}) \). The points from \( P \setminus L \) will be called proper, the points of \( L \) improper or directions. A projective plane \( \pi = (P, L) \) is said to be an \( L \)-transitive plane if the group of all translations of \( \pi(L) \) transitively operates on the set of all points of \( \pi(L) \). Let \( u, v \) be affine lines of \( \pi(L) \) with different directions, then the projective plane \( \pi \) is a \( L \)-transitive plane if and only if the group of all translations of \( \pi(L) \) transitively operates on the lines \( u, v \).

**Proposition 7:** Let \( C = (S, +, \cdot) \) be a generalized Cartesian group and \( \varphi: S \rightarrow S \) a bijection such that \( O^\varphi = 0 \). Then the projective plane \( \pi(S, T(C, \varphi)) \) is a \([\infty]\)-transitive plane if and only if
\[
\forall x, a \in S \exists x' \in S \forall m \in S \quad mx' - Ox' = ma - Oa + O0 = m0 + mx - Ox
\]

**Proof:** I. It suffices to prove that the group of all translations transitively operates on proper points of the line \([0, 0]\). In this case it suffices to show that for each line \([a]\) there exists a translation \( \tau \) such that \([0] = [a] \). Define a mapping \( \tau_a: (x, y) \mapsto (x', (-Ox + Ox + y)^{-1}) \) with \( x' \in S \) uniquely determined by (13) (see (12)). Clearly \( \tau_a \) is bijective. Further it is obvious that the image of
the line \([x]\) is the line \([x']\). Let us consider a line \([m,c]\). If \((x,y) \in [m,c]\), then \(T(\mathcal{C},\varphi)\ (m,x,y) = mx + y^\varphi = c\). Hence it is

\[
T(\mathcal{C},\varphi) = (m,x; (-Ox + Ox + y^\varphi)^{\varphi^{-1}}) = \]

\[(mx - Ox + Ox + y^\varphi) = (ma - Oa + 00 - m0 + mx - Ox) + \]

\[Ox + y^\varphi = (ma - Oa + 00 - m0 + c)\]

or equivalently \((x; (-Ox + Ox + y^\varphi)^{\varphi^{-1}}) \in [m,ma - Oa + 00 - m0 + c]\). If \(x = x'\) for some \(x \in S\), then necessarily \(a = 0\) therefore \(\tau_a = \text{id}\). This implies \(\tau_a\) is a translation. Setting \(x = 0\) in (13), we obtain \(m0 - 00 = ma - Oa\) for each \(m \in S\). Thus suppose \(a \neq 0\). For \(x = 0\) we have \(ma - Oa = ma - Oa + 00 - m0 + mx - Ox\) for each \(m \in S\). Thus suppose \(x \neq 0\). Now choose any element \(k \in S \setminus \{0\}\). By (12) there is \(x'\) such that \(kx - Ox = ka - Oa + 00 - k0 + kx - Ox\).

Further let \(\tau_a\) be a translation for which \((0,0)^{\tau_a} = (a, (-Oa + 00 + 0)^{\varphi^{-1}})\). Then

\[(0,0),(x, (-kx + k0 + 0)^{\varphi^{-1}}) \in [k,ka - Oa + 00 - k0 + kx - Ox]\]

\[(0,0),(a, (-Oa + 00 + 0)^{\varphi^{-1}}) \in [0,00 - 00] \]

\[(a (-Oa + 00 + 0)^{\varphi^{-1}}), (x; (-Ox + Ox - kx + k0 + 0)^{\varphi^{-1}}) \in [k,ka - Oa + 00 - 00]\]

\[(x, (-kx + k0 + 0)^{\varphi^{-1}}), (x; (-Ox + Ox - kx + k0 + 0)^{\varphi^{-1}}) \in [0,00 - kx - k0 + 00]\]

Thus, \((x, (-kx + k0 + 0)^{\varphi^{-1}})^{\tau_a} = (x; (-Ox + Ox - kx + k0 + 0)^{\varphi^{-1}})\) hence \([x]^{\tau_a} = [x']\). For \(m = 0\) is

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Thus let be $m \in S \setminus \{0\}$. Then
\[(x, (-mX + mO + O^\varphi)^{-1}) \in [0, OX - mX + mO + O^\varphi],
\]
\[(x', (-OX + OX - mX + mO + O^\varphi)^{-1}) \in [0, OX - mX + mO + O^\varphi] \cap
\]
\[\cap \{x\}. \text{ Thus, } (x, (-mO + mO + O^\varphi)^{-1}) \in
\]
\[= (x, (-OX + OX - mX + mO + O^\varphi)^{-1}).
\]
But $T(\mathcal{G}, \varphi)(m, O, 0) = mO + O^\varphi =
\]
\[= T(\mathcal{G}, \varphi)(m, X, (-mX + mO + O^\varphi)^{-1}) \text{ and then it follows}
\]

necessarily $T(\mathcal{G}, \varphi)(m, a, (-Oa + O0 + O^\varphi)^{-1}) =
\]
\[= T(\mathcal{G}, \varphi)(m, X, (-OX + OX - mX + mO + O^\varphi)^{-1}), \text{ hence}
\]
\[\text{ma} - Oa + O0 + O^\varphi = mx' - OX + OX - mX + mO + O^\varphi \text{ consequenti-
}\]
\[\text{ly } mx' - OX + ma - Oa + O0 - mO + mx - Ox.
\]
Thus Proposition 7 is proved.

Corollary 7.1: Let $(S, +, \cdot)$ be a generalized Cartesian group such that the condition (13) holds. Then the group $(S, +)$ is Abelian.

Proof: The proof of the preceding corollary depends on the obvious fact that the group of all translations of a $\omega$-transitive plane is Abelian.

Proposition 8: Let $\mathcal{G} = (S, +, \cdot)$ be a generalized Cartesian group such that there exists $e \in S$ where for each $x \in S$ $e \cdot x = e \cdot O$. Further let $\varphi: S \rightarrow S$ be a bijection such that $O^\varphi = O$. Then the projective plane
\[\pi(S, T(\mathcal{G}, \varphi)) \text{ is a } \omega \text{-transitive plane if and only if}
\]
\[(14) \forall x, a \in S \exists x' \in S \forall m \in S mx' - mx = ma - mO.
\]

Proof: I. First we note that by (12) for every $a \in S \setminus \{e\}$ and for every $b \in S$ there exists exactly one $x \in S$ such that $ax - ex = b - eO$ it holds if and only if $ax - eO = b - eO$, ...
ax = b. This implies that for each a ∈ S \ {e} and for each b ∈ S there exists exactly one x ∈ S such that ax = b. Define a mapping \( \tau_a: (x,y) \mapsto (x,ay) \) with x ∈ S uniquely determined by (14). Clearly \( \tau_a \) is a bijective. Further it is obvious that the image of the line \([x]\) is the line \([x']\). Let us consider a line \([m,c]\). If \((x,y) \in [m,c]\), then \(T(C,\varphi)(m,x,y) = mx + y\varphi = c\). Hence it is \(T(C,\varphi)(m,x,y) = mx' + y\varphi = ma - m0 + mx + y\varphi = ma - m0 + c\) or equivalently \((x,y) \in [m,ma - m0 + c]\). If \(x' = x\) for some \(x \in S\) then by (14) \(a = 0\), \(\tau_a = \text{id}\). This implies \(\tau_a\) is a translation. Setting \(x = 0\) in (14), we have \(m0 = ma\) for each \(m \in S\) hence \(0) \varphi = \varphi a\) and consequently \(\varphi (S,T(C,\varphi))\) is a \([\infty]\)-transitive plane.

II. Let \(\varphi (S,T(C,\varphi))\) be a \([\infty]\)-transitive plane. Setting \(m = e\) in (13), we obtain \(ex - Ox = ea - Oa + O0 - e0 + ex - Ox\) then \(-Ox = -Oa + O0 - Ox\) hence \(mx - Ox = mx - Oa + O0 - Ox = ma - Oa + O0 - m0 + mx - Ox\) for each \(m \in S\) and by Corollary 7.1 \(mx = ma - m0 + mx\) therefore \(mx - mx = ma - m0\).

Theorem 1 and Proposition 7 now imply

**Theorem 2:** Let \((S,T)\) be a JTR. Then the projective plane \(\varphi (S,T)\) is a \([\infty]\)-transitive plane if and only if

(i) \(C:=(S,+_{+1}^\star)\) is a generalized Cartesian group

(ii) there exists a bijection \(\varphi: S \rightarrow S\) such that \(0\varphi = 0\), \(T = T(C,\varphi)\)

(iii) \(\forall x,a \in S \exists x' \in S \forall m \in S\)

\[m^*_1 x - l_0^* x = m^*_1 a - l_0^* a + l_0^* l - l m^*_1 O + l m^*_1 x - l O^* l x\]
References


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