MULTIVALUED GENERALIZED CONTRACTIONS AND FIXED POINT THEOREMS

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Abstract: We prove fixed point theorems for multivalued generalized contraction and contractive mappings in metrically convex metric spaces. Theorem 1 generalizes a fixed point theorem of Assad-Kirk for multivalued contraction mappings, Theorem 2 that of Assad for multivalued contractive mappings.

Key words: Multivalued generalized contraction (contractive) mapping, metrically convex metric space.

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1. Introduction. Recently fixed point theorems for multivalued contraction or contractive mappings were obtained by Nadler [9], Assad-Kirk [1] and Assad [2], etc. On the other hand, Kannan [5] initiated studies of certain type of mappings which have many similarities to contraction and nonexpansive mappings. His ideas were further studied and generalized by Reich [10], Ćirić [3], Kannan [8], Hardy-Rogers [5], Goebel-Kirk-Shimi [4] and Wong [11, 12, 13], etc.

In this paper we shall give fixed point theorems for multivalued generalized contraction mappings and generalized contractive mappings. Theorem 1 is an extension of a theorem in Assad-Kirk [1]. Theorem 2 extends a fixed point theorem in Assad [2].
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2. Preliminaries. Let \((X, d)\) be a metric space. For any \(x \in X\) and \(A \subseteq X\), we denote \(d(x, A) = \inf \{d(x, y) : y \in A\}\). It can easily be checked the following lemma.

**Lemma 1.** For any \(x, y \in X\) and \(A \subseteq X\), we have
\[
|d(x, A) - d(y, A)| \leq d(x, y).
\]

Let \(\mathcal{CB}(X)\) denote the family of all nonempty closed bounded subsets of \(X\) and \(D\) be the Hausdorff metric on \(\mathcal{CB}(X)\) induced by the metric \(d\) on \(X\). The following lemmas are direct consequences of the definition of Hausdorff metric.

**Lemma 2.** If \(A, B \in \mathcal{CB}(X)\) and \(x \in A\), then for any positive number \(\varepsilon\), there exists \(y \in B\) such that
\[
d(x, y) \leq D(A, B) + \varepsilon.
\]

**Lemma 3.** For any \(x \in X\) and any \(A, B \in \mathcal{CB}(X)\), it follows that
\[
|d(x, A) - d(x, B)| \leq D(A, B).
\]

\((X, d)\) is said to be metrically convex if for any \(x, y \in X\) with \(x \neq y\), there exists an element \(z \in X\) such that
\[
d(x, z) + d(z, y) = d(x, y).
\]

In Assad and Kirk [11] the following is noted.

**Lemma 4.** If \(K\) is a nonempty closed subset of a complete and metrically convex metric space \((X, d)\), then for any \(x \in K, y \notin K\), there exists a \(z \in \partial K\) (the boundary of \(K\)) such
that
\[ d(x, z) + d(z, y) = d(x, y). \]

3. Generalized contraction mappings. Let \( K \) be a nonempty closed subset of a metric space \((X, d)\) and \( T \) be a mapping of \( K \) into \( \mathcal{C}(X) \). \( T \) is said to be a generalized contraction mapping if there exist nonnegative real numbers \( \alpha, \beta, \gamma \) with \( \alpha + 2\beta + 2\gamma < 1 \) such that for any \( x, y \in K \),
\[
D(T(x), T(y)) \leq \alpha d(x, y) + \beta \{d(x, T(x)) + d(y, T(y))\} + \gamma \{d(x, T(x)) + d(y, T(x))\}.
\]
If \( \beta = \gamma = 0 \), then \( T \) is called \( \alpha \)-contraction.

The following theorem holds.

**Theorem 1.** Let \((X, d)\) be a complete and metrically convex metric space, \( K \) a nonempty closed subset of \( X \). Let \( T \) be a generalized contraction mapping of \( K \) into \( \mathcal{C}(X) \). If for any \( x \in \partial K \), \( T(x) \in K \) and \( \frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} < 1 \), then there is a \( z \in K \) such that \( z \in T(z) \).

**Proof.** Denote \( k = \frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} \), then \( 0 \leq k < 1 \).
If \( k = 0 \), then the conclusion of Theorem 1 is obvious. So we may assume that \( k > 0 \). We choose sequences \( \{x_n\} \) in \( K \) and \( \{y_n\} \) in \( X \) in the following way. Let \( x_0 \in \partial K \) and \( x_1 = y_1 \in T(x_0) \). By Lemma 2, there exists a \( y_2 \in T(x_1) \) such that
\[
d(y_1, y_2) \leq D(T(x_0), T(x_1)) + \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k.
\]
If \( y_2 \in K \), let \( x_2 = y_2 \). If \( y_2 \notin K \), choose an element \( x_2 \in K \) such that \( d(x_1, x_2) + d(x_2, y_2) = d(x_1, y_2) \) using Lemma 4. By induction, we can obtain sequences \( \{x_n\} \), \( \{y_n\} \) such that for
n = 1, 2, \ldots;

(1) \ y_{n+1} \in T(x_n),

(2) \ d(y_n, y_{n+1}) \leq D(T(x_{n-1}), T(x_n)) + \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n,

where

(3) \ y_{n+1} = x_{n+1} \text{ if } y_{n+1} \in K, \text{ or }

(4) \ d(x_n, x_{n+1}) + d(x_{n+1}, y_{n+1}) = d(x_n, y_{n+1}) \text{ if } y_{n+1} \notin K.

We shall estimate the distance \( d(x_n, x_{n+1}) \) for \( n \geq 2 \).

There arise three cases.

(i) The case that \( x_n = y_n \) and \( x_{n+1} = y_{n+1} \). We have

\[
\begin{align*}
\text{d}(x_n, x_{n+1}) &= \text{d}(y_n, y_{n+1}) \\
&\leq D(T(x_{n-1}), T(x_n)) + \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n \\
&\leq \alpha d(x_{n-1}, x_n) + \beta \delta d(x_{n-1}, T(x_{n-1})) + d(x_n, T(x_n)) \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n \\
&+ \gamma \{ d(x_{n-1}, T(x_n)) + d(x_n, T(x_{n-1})) \} \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n \\
&\leq \alpha d(x_{n-1}, x_n) + \beta \delta d(x_{n-1}, x_n) + d(x_n, x_{n+1}) \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n,
\end{align*}
\]

hence

\[
(1 - \beta - \gamma)d(x_n, x_{n+1}) \leq (\alpha + \beta + \gamma)d(x_{n-1}, x_n) + \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n
\]

and

\[
\begin{align*}
d(x_n, x_{n+1}) &\leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(x_{n-1}, x_n) + \frac{k^n}{1 + \beta + \gamma}.
\end{align*}
\]

(ii) The case that \( x_n = y_n \) and \( x_{n+1} \neq y_{n+1} \). By (4) we obtain that

\[
\begin{align*}
d(x_n, x_{n+1}) &\leq d(x_n, y_{n+1}) = d(y_n, y_{n+1}).
\end{align*}
\]

As in the case (i), we have
\[ d(y_n, y_{n+1}) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(x_{n-1}, x_n) + \frac{k^n}{1 + \beta + \gamma}, \]

thus

\[ d(x_n, x_{n+1}) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(x_{n-1}, x_n) + \frac{k^n}{1 + \beta + \gamma}. \]

(iii) The case that \( x_n \neq y_n \) and \( x_{n+1} = y_{n+1} \). In this case \( x_{n-1} = y_{n-1} \) holds. We have

\[ d(x_n, x_{n+1}) = d(x_n, y_n) + d(y_n, x_{n+1}) = d(x_n, y_n) + d(y_n, y_{n+1}). \]

By (2) it follows that

\[ d(y_n, y_{n+1}) \leq D(T(x_{n-1}), T(x_n)) + \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n \]

\[ \leq \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, T(x_{n-1})) + d(x_n, T(x_n)) \]

\[ + \gamma d(x_{n-1}, T(x_n)) + d(x_n, T(x_n)) = \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n \]

\[ \leq \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, y_n) + d(x_n, x_{n+1}) \]

\[ + \gamma d(x_{n-1}, x_n) + d(x_n, x_{n+1}) = \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n. \]

Since \( 0 \leq \alpha < 1 \) and \( d(x_{n-1}, x_n) + d(x_n, y_n) = d(x_{n-1}, y_n) \), we obtain

\[ d(x_n, x_{n+1}) \leq (1 + \gamma) d(x_n, y_n) + (\alpha + \gamma) d(x_{n-1}, x_n) + \]

\[ + \beta d(x_{n-1}, y_n) + (\beta + \gamma) d(x_n, x_{n+1}) + \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n \]

\[ \leq (1 + \gamma) d(x_{n-1}, y_n) + \beta d(x_{n-1}, y_n) \]

\[ + (\beta + \gamma) d(x_n, x_{n+1}) + \frac{1 - \beta - \gamma}{1 + \beta + \gamma} k^n, \]

and

\[ d(x_n, x_{n+1}) \leq \frac{1 - \beta - \gamma}{1 + \beta + \gamma} \left( d(x_{n-1}, y_n) + \frac{k^n}{1 + \beta + \gamma} \right). \]

As in the case (ii), we have

\[ d(x_{n-1}, y_n) \leq \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} d(x_{n-2}, x_{n-1}) + \frac{k^{n-1}}{1 + \beta + \gamma}. \]
Thus it follows that
\[
d(x_n, x_{n+1}) \leq \frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} \ d(x_{n-2}, x_{n-1})
\]
\[+ \frac{k^{n-1}}{1 - \beta - \gamma} + \frac{k^n}{1 + \beta + \gamma}.
\]

The case that \( x_n + y_n \) and \( x_{n+1} + y_{n+1} \) does not occur. Since
\[
\frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} \leq \frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2}, \text{ for } n \geq 2 \text{ we have}
\]
\[
d(x_n, x_{n+1}) \leq \begin{cases} 
kd(x_{n-1}, x_n) + \frac{k^n}{1 - \beta - \gamma}, & \text{or} \\
kd(x_{n-2}, x_{n-1}) + \frac{k^{n-1} + k^n}{1 - \beta - \gamma}.
\end{cases}
\]

Put \( \sigma = \frac{1}{k^2} \max \{ \| x_0 - x_1 \|, \| x_1 - x_2 \| \} \), then by induction we can show that
\[
d(x_n, x_{n+1}) \leq k^{\frac{n}{2}} (\sigma + \frac{n}{1 - \beta - \gamma}) (n = 1, 2, \ldots).
\]
It follows that for any \( m > n \geq 1 \),
\[
d(x_n, x_m) \leq \sigma \sum_{i=n}^{m-1} (k^2)^i + \frac{1}{1 - \beta - \gamma} \sum_{i=n}^{m-1} i(k^2)^i.
\]
This implies that \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete and \( K \) is closed, \( \{x_n\} \) converges to some point \( z \in K \). By the way of choosing \( \{x_n\} \), there exists a subsequence \( \{x_{n_i}\} \) of \( \{x_n\} \) such that \( x_{n_i} = y_{n_i} \) (\( i = 1, 2, \ldots \)). Then we have
\[
d(x_{n_i}, T(z)) \leq D(T(x_{n_i-1}), T(z))
\]
\[\leq \alpha d(x_{n_i-1}, z) + \beta \left( d(x_{n_i-1}, T(x_{n_i-1})) + d(z, T(z)) \right)^\gamma
\]
\[+ \gamma \left( d(x_{n_i-1}, T(z)) + d(z, T(x_{n_i-1})) \right)^\gamma
\]
\[\leq \alpha d(x_{n_i-1}, x_{n_i}) + d(x_{n_i}, z) + \beta d(x_{n_i-1}, x_{n_i})
\]
Therefore, $d(x_n, T(z)) \to 0$ as $i \to \infty$. By the inequality
\[
d(z, T(z)) \leq d(x_n, z) + d(x_n, T(z))
\]
and the above result, it follows that $d(z, T(z)) = 0$. Since $T(z)$ is closed, this implies that $z \in T(z)$. q.e.d.

Since every Banach space is metrically convex, we have the following corollary for singlevalued mappings.

**Corollary 1.** Let $E$ be a Banach space and $K$ be a nonempty closed subset of $E$. Let $f$ be a generalized contraction mapping of $K$ into $E$. If $f(\partial K) \subset K$ and
\[
\frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} < 1,
\]
then there exists a (unique) fixed point of $f$ in $K$.

3. **Generalized contractive mappings.** Let $K$ be a nonempty closed subset of a metric space $(X, d)$. Let $T$ be a mapping of $K$ into $\mathcal{CB}(X)$. $T$ is said to be a generalized contractive mapping if there exist nonnegative real numbers $\alpha$, $\beta$, $\gamma$ such that for any $x, y \in K$ with $x \neq y$,
\[
D(T(x), T(y)) < \alpha d(x, y) + \beta \{d(x, T(x)) + d(y, T(y))\}
\]
\[
+ \gamma \{d(x, T(y)) + d(y, T(x))\},
\]
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where $0 < \alpha + 2\beta + 2\gamma \leq 1$. If $\beta = \gamma = 0$ and $\alpha = 1$, then $T$ is called contractive. $T$ is said to be continuous at $x_0 \in K$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$D(T(x), T(x_0)) < \varepsilon$$

whenever $d(x, x_0) < \delta$. If $T$ is continuous at each point of $K$, we say that $T$ is continuous on $K$.

We shall give a fixed point theorem for continuous generalized contractive mappings.

Theorem 2. Let $(X, d)$ be a complete and metrically convex metric space and $K$ be a nonempty compact subset of $X$. Let $T$ be a generalized contractive mapping of $K$ into $\mathcal{B}(X)$ and continuous on $K$. If for any $x \in \partial K$, $T(x) \subset K$ and

$$\frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} \leq 1,$$

then there exists an element $z \in K$ such that $z \in T(z)$.

Proof. Define a function $g$ of $K$ into $\mathbb{R}^+$ (nonnegative real numbers) by $g(x) = d(x, T(x))$ ($x \in K$), then by Lemma 1 and Lemma 3, we have

$$|g(x) - g(y)| \leq d(x, T(x)) - d(y, T(x)) + D(T(x), T(y)) = d(x, y) + D(T(x), T(y)).$$

Hence $g$ is continuous and since $K$ is compact, there exists a $z \in K$ such that $g(z) = \min \{g(x) : x \in K\}$. Suppose that $g(z) = 0$, then we obtain a contradiction. For each $n = 1, 2, \ldots$, there exists a $x_n \in T(z)$ for which

$$d(x_n, z) \leq g(z) + \frac{1}{n}.$$

If $x_n \in K$ for $n$ sufficiently large, then some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges to an $x_0 \in K$. We may assume that $x_0 \neq z$, then
\[ g(x_0) = d(x_0, T(x_0)) \leq D(T(z), T(x_0)) \]
\[ < \alpha d(z, x_0) + \beta d(z, T(z)) + d(x_0, T(x_0)) \]
\[ \cdot \gamma d(z, T(x_0)) + d(x_0, T(z)) \]
\[ \leq \alpha g(z) + \beta g(x_0) + \gamma g(z) + g(x_0) \]

and

\[ (1 - \beta - \gamma) g(x_0) < (\alpha + \beta + \gamma) g(z). \]

Thus

\[ g(x_0) < \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} g(z) \leq g(z), \]

contradicting the minimality of \( g(z) \). If there exists a subsequence \( \{x_{n_1}\} \) of \( \{x_n\} \) such that \( x_{n_1} \notin K \), then \( z \notin \partial K \). For simplicity, we may assume that \( x_n \notin K, \ n = 1, 2, \ldots \). By Lemma 4, for each \( n \) there exists \( y_n \in \partial K \) for which

\[ d(x_n, y_n) + d(y_n, z) = d(x_n, z). \]

Since \( K \) is compact and \( T(y_n) \subseteq K \), there exists \( w_n \in T(y_n) \) such that \( d(x_n, w_n) = d(x_n, T(y_n)) \). We may also assume that \( \{y_n\} \) converges to some \( y_0 \in \partial K \). Let

\[ \varepsilon = \alpha d(y_0, z) + \beta d(y_0, T(y_0)) + d(z, T(z)) \]
\[ + \gamma d(y_0, T(z)) + d(z, T(y_0)) \cdot D(T(y_0), T(z)), \]

then \( \varepsilon > 0 \), because \( y_0 \notin z \). For this \( \varepsilon \), there exists a positive integer \( N \) such that for any \( n \geq N \)

\[ d(y_0, z) - d(y_n, z) < 2\varepsilon, \]
\[ g(y_0) - \varepsilon < g(y_n), \]
\[ d(x_n, z) < g(z) + 2\varepsilon, \] and
\[ D(T(y_n), T(z)) < D(T(y_0), T(z)) + 2\varepsilon. \]

Then for any \( n \geq N \), we have

\[ g(y_0) - \varepsilon < g(y_n) = d(y_n, T(y_n)) \]

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\[
\sum d(y_n, w_n) \leq d(y_n, x_n) + d(x_n, w_n) = d(x_n, y_n) + d(x_n, T(y_n)) \\
\leq d(x_n, y_n) + d(T(z), T(y_n)) \leq d(x_n, y_n) + d(T(z), T(y_0)) + 2\varepsilon \\
= d(x_n, y_n) + \alpha d(y_o, z) + \beta \delta d(y_o, T(y_0)) + d(z, T(z)) + \\
\gamma \delta d(y_o, T(z)) + d(z, T(y_o)) - 6\varepsilon \\
\leq d(x_n, y_n) + (\alpha + 2\gamma) d(y_o, z) + (\beta + \gamma) g(y_o) + (\beta + \gamma) g(z) - \\
- 6\varepsilon < (1 + \beta + \gamma) g(z) + (\beta + \gamma) g(y_o) - 2\varepsilon ,
\]
hence
\[
g(y_o) < \frac{1 + \beta + \gamma}{1 - \beta - \gamma} g(z) - \frac{\varepsilon}{1 - \beta - \gamma}
\]
Take a \( u \in T(y_o) \) such that \( d(y_o, T(y_o)) = d(y_o, u) \). Since \( g(z) > 0 \), \( u \neq y_o \). Thus we obtain
\[
g(u) = d(u, T(u)) \leq d(T(y_o), T(u)) \\
< \alpha d(y_o, u) + \beta \delta d(y_o, T(y_o)) + d(u, T(u)) + \\
\gamma \delta d(y_o, T(u)) + d(u, T(y_o)) + \\
\leq (\alpha + \beta + \gamma) g(y_o) + (\beta + \gamma) g(u)
\]
and
\[
g(u) < \frac{\alpha + \beta + \gamma}{1 - \beta - \gamma} g(y_o).
\]
Therefore it follows that
\[
g(u) < \frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} g(z) - \frac{(\alpha + \beta + \gamma)\varepsilon}{(1 - \beta - \gamma)^2}
\]
\[
\leq g(z) - \frac{(\alpha + \beta + \gamma)\varepsilon}{(1 - \beta - \gamma)^2}.
\]
This is a contradiction. Hence \( g(z) = 0 \) and since \( T(z) \) is closed, we have \( z \in T(z) \). \ q.e.d.

In Banach spaces, the following corollary holds.
Corollary 2. Let $K$ be a nonempty compact subset of a Banach space $E$ and $f$ be a continuous generalized contractive mapping of $K$ into $E$. If $f(\partial K) \subset K$ and 
\[
\frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} \leq 1,
\]
then there exists a (unique) fixed point of $f$ in $K$.

Remark. If for any $x \in K$, $T(x) \subset K$ in Theorem 1 (or Theorem 2), then the conditions that $k = \frac{(\alpha + \beta + \gamma)(1 + \beta + \gamma)}{(1 - \beta - \gamma)^2} < 1$ (or $k \leq 1$) and that $X$ is metrically convex are unnecessary.

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