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# EPIMORPHISMS IN SOME GROUPOID VARIETIES

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**Abstract**: Two classes of groupoid identities generating varieties with non-surjective epimorphisms are investigated.

**Key words**: Epimorphism, groupoid, variety.

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Every variety of universal algebras can be viewed as a category of structures. In this case, a morphism is a monomorphism iff it is an injective homomorphism. The corresponding assertion for epimorphisms is not true. The first known examples of varieties with non-surjective epimorphisms seem to be the varieties of semigroups and rings. The reader is referred to [1] for original proofs of these facts. The situation in semigroups was investigated in detail in [2] and [3]. Some generalizations for algebras and categories were proved in [4] and [5]. In this paper we deal with two methods which enable us to find a large number of groupoid identities generating varieties with non-surjective epimorphisms. The first one is in some sense a generalization of the classical method used for commutative semigroups. The corresponding identities are similar to the medial law $xy.uv = xu.yv$. The second method can be used for certain varieties of commutative groupoids, namely for those varieties, every groupoid of which has at most one idempotent.

1. <u>Introduction.</u> Let $F$ be an absolutely free groupoid generated by a set $X$ of variables. Elements from $F$ are called (groupoid) terms. We define the length $l(t)$ of a term $t$ by $l(u) = 1$ for every $u \in X$ and $l(rs) = l(r) + l(s)$ for all $r, s \in F$. Further we denote by $var(t)$ the set of all variables occurring in $t$. The notation $t = t(x_1, \dots, x_n)$ means that $var(t) = \{x_1, \dots, x_n\}$. If $t$ is a term and $u$ is a variable then $o(t,u)$ is the number of occurrences of $u$ in $t$. If $G$ is a groupoid then $t_G$ is the corresponding n-ary operation defined on $G$ by means of the term $t$. If $t$, $s$ are terms then $Mod(t \mathrel{\hat{=}} s)$ is the variety of groupoids satisfying the identity $t \mathrel{\hat{=}} s$. We put $\mathcal{C} = Mod(xy \mathrel{\hat{=}} yx)$, $\mathcal{I} = Mod(x \mathrel{\hat{=}} xx)$, $\mathcal{S} = Mod(x.yz \mathrel{\hat{=}} xy.z)$, $\mathcal{M} = Mod(xy.uv \mathrel{\hat{=}} xu.yv)$, $\mathcal{D} = Mod(x.yz \mathrel{\hat{=}} xy.xz,\ zy.x \mathrel{\hat{=}} zx.yx)$. The following lemma is clear.

1.1. <u>Lemma</u>. $\mathcal{C} \cap \mathcal{S} \subseteq \mathcal{M}$ and $\mathcal{M} \cap \mathcal{I} \subseteq \mathcal{D}$.

A groupoid identity $t \mathrel{\hat{=}} s$ is said to be quasibalanced if $o(t,u) = o(s,u)$ for every variable $u$. A groupoid variety is called quasibalanced if it can be determined by a set of quasibalanced identities. The following lemma is obvious.

1.2. <u>Lemma</u>. The following conditions are equivalent for a groupoid variety $\mathcal{U}$:

(i) If $\mathcal{U} \subseteq Mod(t \mathrel{\hat{=}} s)$ then $t \mathrel{\hat{=}} s$ is quasibalanced.

(ii) $\mathcal{U}$ is quasibalanced.

(iii) $\mathcal{C} \cap \mathcal{S} \subseteq \mathcal{U}$.

2. Closed subgroupoids. Let G be a groupoid and $a \in G$. We define two mappings $L_a, R_a$ of G into G by $L_a(b) = ab$ and $R_a(b) = ba$ for every $b \in G$. The groupoid G is called left(right) cancellation(division) groupoid if $L_a$ ($R_a$) is an injective(surjective)mapping for every $a \in G$. Further, G is called a left (right) quasigroup if $L_a$ ($R_a$) is bijective for every $a \in G$. Finally, G is a cancellation groupoid if it is both left and right cancellation groupoid. Similarly we define division groupoids and quasigroups.

Let H be a subgroupoid of a groupoid G. We say that H is a left closed subgroupoid of G if $b \in H$ whenever $a, b \in G$ and $a, ab \in H$. Similarly we define right closed and closed subgroupoids. If $M \subseteq G$ is a subset then $cl_G(M)$ denotes the left closed subgroupoid generated by M. Similarly we define $cr_G(M)$ and $c_G(M)$. A subgroupoid $K \subseteq G$ is called left dense if $cl_G(K) = G$. Similarly we define right dense and dense subgroupoids. The following two lemmas are easy.

2.1. Lemma. Let H be a subgroupoid of a groupoid G. Then H is a left dense (resp. right dense, dense) subgroupoid of $cl_G(H)$ (resp. $cr_G(H)$, $c_G(H)$).

2.2. Lemma. Let H be a left (right) closed subgroupoid of a left (right) division groupoid G. Then H is a left (right) division groupoid.

2.3. Lemma. Let H be a left (right) dense subgroupoid of a groupoid G and f, g be two homomorphisms of G into a left (right) cancellation groupoid K such that $f \mid H = g \mid H$. Then $f = g$.

*Proof*. Put $A = \{a \in G \mid f(a) = g(a)\}$. Then $H \subseteq A$ and $A$ is a subgroupoid of $G$. Moreover, $A$ is a left right closed subgroupoid, as one may check easily. Hence $A = G$.

2.4. *Lemma*. Let $H$ be a dense subgroupoid of a groupoid $G$ and $f$, $g$ be two homomorphisms of $G$ into a cancellation groupoid $K$ such that $f \mid H = g \mid H$. Then $f = g$.

*Proof*. Similar to that of 2.3.

A groupoid $G$ is said to be an LN-groupoid (RN-groupoid) if every factorgroupoid of the cartesian product $G \times G$ is a left (right) cancellation groupoid. Further, $G$ is an N-groupoid if it is both an LN and RN-groupoid. The following result is not difficult.

2.5. *Lemma*. (i) Every group is an N-quasigroup.

(ii) Every quasigroup from $\mathcal{C} \cap \mathcal{D}$ is an N-quasigroup.

The class of quasigroups can be considered as a variety of algebras with three binary operations. The following lemma is evident.

2.6. *Lemma*. Let $G$ be a subgroupoid of a quasigroup $Q$. Then $G$ is a dense subgroupoid of $Q$ iff $Q$ is generated by $G$ as a quasigroup.

3. *Medial groupoids and generalizations*. Let $t = t(x_1,\dots,x_n)$ be a term. We put

$$\mathcal{V}(t) = \mathrm{Mod}\ (t(x_1 y_1,\dots,x_n y_n) \hat{=} t(x_1,\dots,x_n).t(y_1,\dots,y_n)).$$

For example, if $t = x.yx$ then

$$\mathcal{V}(t) = \mathrm{Mod}\ (x_1 y_1.(x_2 y_2.x_1 y_1) \hat{=} (x_1.x_2 x_1)(y_1.y_2 y_1)).$$

3.1. *Lemma*. $\mathcal{M} = \mathcal{V}(xy)$.

*Proof*. Easy.

3.2. *Lemma*. $\mathcal{M} \subseteq \mathcal{V}(t)$ for every term $t$.

Proof. By induction on $l(t)$.

3.3. Lemma. Let t be a term. Then Mod $(x \hat{=} t) \subseteq \mathcal{V}(t)$.

Proof. Easy.

Let $t = t(x,y)$ be a term and G be a groupoid. We shall say that G is a t-complete groupoid if for all a, $b \in G$ there are c, $d \in G$ such that $t_G(a,c) = b = t_G(d,a)$. The following lemma is clear.

3.4. Lemma. Let $t = xy$ and G be a groupoid. Then G is t-complete iff G is a division groupoid.

Let R(+) be the additive group of rational numbers, P be the set of positive rational numbers and $a \circ b = 1/2(a + b)$ for all a, $b \in R$. The next lemma is almost obvious.

3.5. Lemma. (i) $R(+) \in \mathcal{C} \cap \mathcal{S}$ , R(+) is an N-quasigroup and P(+) is a dense subgroupoid of R(+).

(ii) $R(\circ) \in \mathcal{M} \cap \mathcal{C} \cap \mathcal{I}$ , $R(\circ)$ is an N-quasigroup and $P(\circ)$ is a dense subgroupoid of $R(\circ)$.

(iii) R(+), $R(\circ) \in \mathcal{V}(t)$ for every term t.

(iv) R(+), $R(\circ)$ are t-complete for every term $t = t(x,y)$.

3.6. Lemma. Let $t = t(x,y)$ be a term and K, H be two subgroupoids of a groupoid $G \in \mathcal{V}(t)$. Suppose that K, H are t-complete, $K \cap H$ is non-empty and G is generated by $K \cup H$. Then G is a homomorphic image of the cartesian product $K \times H$.

Proof. Define $f: K \times H \longrightarrow G$ by $f(a,b) = t_G(a,b)$ for all $a \in K$ and $b \in H$. Since $G \in \mathcal{V}(t)$, f is a homomorphism. Let $a \in K \cap H$ and $b \in H$ be arbitrary. There is $c \in H$ such that $b = t_H(a,c)$. However $t_H(a,c) = t_G(a,c) = f(a,c)$. Hence $H \subseteq \mathrm{Im}\, f$. Similarly $K \subseteq \mathrm{Im}\, f$ and $\mathrm{Im}\, f = G$.

3.7. Proposition. Let $t = t(x,y)$ be a term and $G \in \mathcal{V}(t)$ be a t-complete LN-groupoid (RN-groupoid). Let $H \subseteq G$ be a left (right) dense subgroupoid. Then the inclusion $H \subseteq G$ is an epimorphism in $\mathcal{V}(t)$.

Proof. Let $f, g: G \longrightarrow K$ be such that $K \in \mathcal{V}(t)$ and $f \restriction H = g \restriction H$. We can assume that K is generated by $A \cup B$, where $A = \operatorname{Im} f$ and $B = \operatorname{Im} g$. The groupoids A, B are homomorphic images of G, and therefore A, B are t-complete. Further, $f(H) = g(H) \subseteq A \cap B$. By 3.6, K is a homomorphic image of $A \times B$. However $A \times B$ is a homomorphic image of $G \times G$, and consequently K is a left (right) cancellation groupoid. An application of 2.3 finishes the proof.

3.8. Proposition. Let $t = t(x,y)$ be a term and $G \in \mathcal{V}(t)$ be a t-complete N-groupoid. Let $H \subseteq G$ be a dense subgroupoid. Then the inclusion $H \subseteq G$ is an epimorphism in $\mathcal{V}(t)$.

Proof. Similar to that of 3.7.

3.9. Corollary. Let Q be a medial N-quasigroup generated as a quasigroup by a subgroupoid G. Then the inclusion $G \subseteq Q$ is an epimorphism in the variety $\mathcal{M}$.

Proof. Apply 3.8, 3.1, 3.4 and 2.6.

3.10. Theorem. Let t be a groupoid term containing at least two variables. The following varieties have non-surjective epimorphisms:

(i) Every variety $\mathcal{U}$ such that $\mathcal{C} \cap \mathcal{S} \subseteq \mathcal{U} \subseteq \mathcal{V}(t)$.

(ii) Every variety $\mathcal{U}$ such that $\mathcal{M} \cap \mathcal{C} \cap \mathcal{I} \subseteq \mathcal{U} \subseteq \mathcal{V}(t)$.

(iii) Every variety $\mathcal{U} \cap \mathcal{V}(t)$, where $\mathcal{U}$ is a quasi-balanced variety.

(iv) The variety generated by $\mathcal{C} \cap \mathcal{S}$ and $\operatorname{Mod}(x \triangleq t)$.

Proof. (i) It is easy to see that there exists a term

$s = s(x,y)$ such that $\mathcal{V}(t) \subseteq \mathcal{V}(s)$. According to 3.5 and 3.8, the inclusion $P(+) \subseteq R(+)$ is an epimorphism in $\mathcal{V}(s)$, and hence in $\mathcal{V}(t)$.

(ii) Similarly as for (i).

(iii) and (iv). Clearly, $\mathcal{C} \cap \mathcal{S} \subseteq \mathcal{U} \cap \mathcal{V}(t) \subseteq \mathcal{V}(t)$ and $\mathrm{Mod}\,(x \triangleq t) \subseteq \mathcal{V}(t)$.

Let $\mathcal{U}$ be a groupoid variety. We shall say that $\mathcal{U}$ satisfies the condition (M) if G is a cancellation groupoid, whenever $G \in \mathcal{U}$ and G/r is a quasigroup where r is the least congruence with $G/r \in \mathcal{M}$.

3.11. <u>Proposition</u>. The variety $\mathcal{C} \cap \mathcal{D}$ satisfies (M).

<u>Proof</u>. See [6], Lemma 8.5.

3.12. <u>Proposition</u>. Let a groupoid variety $\mathcal{U}$ satisfy (M) and $Q \in \mathcal{U}$ be an N-quasigroup. Let $G \subseteq Q$ be a dense subgroupoid. Then $G \subseteq Q$ is an epimorphism in $\mathcal{U}$.

<u>Proof</u>. Let $f, g\colon Q \longrightarrow K$, $K \in \mathcal{U}$ and $f \mid G = g \mid G$. We can assume that K is generated by $\mathrm{Im}\, f \cup \mathrm{Im}\, g$. Similarly as in the proof of 3.7, we can show that K/r is a quasigroup where r is the least congruence with $K/r \in \mathcal{M}$ (use 3.4 and 3.1). Hence K is a cancellation groupoid and the rest is clear.

3.13. <u>Corollary</u>. The varieties $\mathcal{M}$, $\mathcal{M} \cap \mathcal{I}$, $\mathcal{M} \cap \mathcal{C}$, $\mathcal{M} \cap \mathcal{S}$, $\mathcal{M} \cap \mathcal{D}$, $\mathcal{C} \cap \mathcal{S}$, $\mathcal{D} \cap \mathcal{C}$, $\mathcal{M} \cap \mathcal{C} \cap \mathcal{I}$, $\mathcal{D} \cap \mathcal{C} \cap \mathcal{I}$, $\mathcal{M} \cap \mathcal{C} \cap \mathcal{D}$ have non-surjective epimorphisms.

4. <u>Several lemmas</u>. Let F (resp. K) be the absolutely free (resp. free commutative) groupoid generated by x. Let $\varphi : F \longrightarrow K$ be the canonical homomorphism. The following three lemmas are easy.

4.1. Lemma. Let a, b, c, $d \in K$ and ab = cd. Then either a = c, b = d or a = d, b = c.

4.2. Lemma. Let a, $b \in F$ and $\varphi(a) = \varphi(b)$. Then $l(a) = l(b)$.

4.3. Lemma. Let a, $b \in F$, $\varphi(a) = \varphi(b)$ and G be a commutative groupoid. Then $a_G = b_G$.

Let $p \in K$, $q \in F$ be such that $\varphi(q) = p$ and G be a commutative groupoid. We put $l(p) = l(q)$ and $p_G = q_G$.

4.4. Lemma. Let p, q, $a \in K$ and $p_K(a) = q_K(a)$. Then $p = q$.

Proof. By induction on $l(p) + l(q)$.

4.5. Lemma. Let p, q, a, $b \in K$. Then $p_K(a) = q_K(b)$ iff at least one of the following conditions holds:

(i) $p = q_K(r)$ and $r_K(a) = b$ for some $r \in K$.

(ii) $q = p_K(r)$ and $r_K(b) = a$ for some $r \in K$.

Proof. The direct implication can be proved easily by 4.4 and induction on $l(p) + l(q)$, while the converse implication is trivial.

An element $p \in K$ is called reducible if $p = q_K(r)$ for some q, $r \in K$, $q \neq x \neq r$. The following lemma is trivial.

4.6. Lemma. Let $p \in K$ be such that $l(p)$ is a prime. Then p is not reducible.

4.7. Lemma. Let p, $q \in K$ be not reducible. Suppose that $p \neq q$ and $p \neq x \neq q$. Then $p_K(a) \neq q_K(b)$ for all a, $b \in K$.

Proof. Use 4.5.

Define a relation $\eta$ on K by $a \, \eta \, b$ iff b = ac for some $c \in K$. Let $\rho$ denote the least reflexive and transitive relation containing $\eta$. If a, $b \in K$ and $a \, \rho \, b$ then we shall say that a is a subterm of b. Finally we shall define symmetric groupoid terms by induction. Every variable is a symmetric term. If t is a symmetric term then tt is symmetric.

5. Commutative groupoids. Let $\mathcal{U}$ be a groupoid variety. Then $\mathcal{T}(\mathcal{U})$ denotes the class of all $G \in \mathcal{U}$ with the following property: If $e \notin G$ then there exists a groupoid $H \in \mathcal{U}$ such that $H = G \cup \{e\}$, e is an idempotent and G is a subgroupoid of H.

5.1. Proposition. Let $\mathcal{U}$ be a groupoid variety such that every groupoid from $\mathcal{U}$ contains at most ine idempotent. Let $H \in \mathcal{U}$ and G be a subgroupoid of H such that $H = G \cup \{e\}$ and e is an idempotent. Then the inclusion $G \subseteq H$ is an epimorphism in $\mathcal{U}$.

Proof. Let $A \in \mathcal{U}$ and f, g be two homomorphisms of H into A such that $f \mid G = g \mid G$. Since e is idempotent, f(e) and g(e) are so, and consequently f(e) = g(e). Thus f = g.

5.2. Corollary. Let $\mathcal{U}$ be a groupoid variety such that every groupoid from $\mathcal{U}$ contains at most one idempotent and $\mathcal{T}(\mathcal{U})$ is non-empty. Then $\mathcal{U}$ has non-surjective epimorphisms.

Let E (resp. F) be the absolutely free groupoid generated by x, y (resp. x). We shall assume that F is a subgroupoid of E. Further, let $t, p, q \in E$ be such that $t, p \in F$ and $\mathrm{var}(q) = \{y\}$. Put $\mathcal{A} = \mathrm{Mod}\,(xy \doteq yx,\ t \doteq pq)$.

5.3. Lemma. Every groupoid from $\mathcal{A}$ contains at most one idempotent.

Proof. Let $G \in \mathcal{A}$ and $a, b \in G$ be idempotents. Then $a = t_G(a) = p_G(a).q_G(b) = ab = ba = t_G(b) = b$.

5.4. Lemma. Let $G \in \mathcal{A}$ and $a, b \in G$ be such that $p_G(a) = p_G(b)$. Then $t_G(a) = t_G(b)$.

Proof. Obvious.

5.5. Lemma. Let $G \in \mathcal{A}$. The following conditions are equivalent:

(i) $G \in \mathcal{T}(\mathcal{A})$.

(ii) $p_G(G) \cap q_G(G)$ is empty.

Proof. (i) implies (ii). Let $G, H \in \mathcal{A}$, $H = G \cup \{e\}$, $ee = e$ and $p_G(a) = q_G(b)$ for some $a, b \in G$. Then $t_G(a) = p_G(a).e = e.q_G(b) = t_H(e) = e$, a contradiction with $e \notin G$.

(ii) implies (i). Let $e \notin G$ and $H = G \cup \{e\}$. Put $a \circ b = ab$, $e \circ e = e$, $e \circ p_G(a) = t_G(a) = p_G(a) \circ e$, $e \circ c = e = c \circ e$ for all $a, b, c \in G$, $c \notin p_G(G)$. As it is easy to see, $G$ is a subgroupoid of $H(\circ)$ and $H(\circ) \in \mathcal{A}$.

5.6. Corollary. Let $t, p, q$ be three groupoid terms such that var $(t) = \{x\} =$ var $(p)$ and var $(q) = \{y\}$. Let $\mathcal{A} = \mathrm{Mod}\,(xy \triangleq yx,\ t \triangleq pq)$ and suppose that there exists a groupoid $G \in \mathcal{A}$ such that $p_G(a) \neq q_G(b)$ for all $a, b \in G$. Then the variety $\mathcal{A}$ has non-surjective epimorphisms.

5.7. Proposition. The variety $\mathcal{A} = \mathrm{Mod}\,(x.xx \triangleq (x.xx)((y.yy)(y.yy)),\ xy \triangleq yx)$ has non-surjective epimorphisms.

Proof. Let $G = \{0,1\}$ and $0.0 = 1$, $1.0 = 0.1 = 1.1 = 0$. One may check easily that $G \in \mathcal{A}$ and $a.aa \neq (b.bb)(b.bb)$ for all $a, b \in G$. Now we can use 5.6.

5.8. Proposition. The variety $\mathcal{A} = \mathrm{Mod}\,(xy \triangleq yx,\ x \triangleq (xx)(y.yy))$ has non-surjective epimorphisms.

Proof. Let $K$ be the free commutative groupoid generated by x and M be the set of all $p \in K$ such that non $(aa)(x.xx) \wp p$ and non $b.bb \wp p$ for all $a, b \in K$, $b \neq x$. If $a, b \in M$ and $ab \in M$ then we put $a \circ b = ab$. Further we put $aa \circ x.xx = a = x.xx \circ aa$ and $a \circ aa = x.xx = aa \circ a$ for every $a \in M$. We have defined a groupoid $M(\circ)$ and $M(\circ) \in \mathcal{O}\!\iota$, as one may verify easily. Clearly, $a \circ a \neq b \circ (b \circ b)$ for all $a, b \in M$. Now we can use 5.6.

Let K be the free commutative groupoid generated by x and t, $p, q \in K$ be three elements satisfying the following conditions:

(1) p, q are not reducible.

(2) $p \neq q$.

(3) non $x.q_K(a) \wp p$ for every $a \in K$.

(4) non $x.q_K(a) \wp t$ for every $a \in K$.

(5) non $x.p_K(a) \wp q$ for every $a \in K$.

(6) non $x.p_K(a) \wp t$ for every $a \in K$.

(7) non $p_K(a).q_K(b) \wp t$ for all $a, b \in K$.

5.9. <u>Lemma</u>. $p \neq x$ and $q \neq x$.

<u>Proof</u>. Let $p = x$. Since $p \neq q$, $q \neq x$ and $l(q) \geq 2$. In particular, $xx = xp$ is a subterm of q, a contradiction. Similarly $q \neq x$.

Let M be the set of all $r \in K$ such that non $p_K(a).q_K(b) \wp r$ for all $a, b \in K$. It is visible that $p, q, t \in M$.

5.10. <u>Lemma</u>. $t_K(a) \in M$ for every $a \in M$.

<u>Proof</u>. We shall prove by induction on $l(k)$ that $k_K(a) \in M$ for every subterm k of t. If $k = x$ then there is nothing to prove. Let $k = bc$, $b_K(a) \in M$ and $c_K(a) \in M$. If $b_K(a).c_K(a) \in M$ then $k_K(a) \in M$. Suppose that $b_K(a).c_K(a) \notin M$. Then there are $d, e \in K$ such that $p_K(d).q_K(e) \wp b_K(a).c_K(a)$. However $b_K(a)$,

$c_K(a) \in M$, and hence $p_K(d).q_K(e) = b_K(a).c_K(a)$. We shall assume that $p_K(d) = b_K(a)$ and $q_K(e) = c_K(a)$ (the other case is similar). Taking into account 4.5, we have the following possibilities:

(i) $b = p_K(r)$ and $c = q_K(s)$ for some $r, s \in K$. Then $p_K(r).q_K(s)$ is a subterm of t, a contradiction.

(ii) $b = p_K(r)$ and $q = c_K(s)$ for some $r, s \in K$. If $c = x$ then $bc = p_K(r).x$ is a subterm of t, a contradiction. Hence $c \neq x$, and so $s = x$, since q is not reducible. Consequently $q = c$ and $bc = p_K(r).q_K(x)$ is a subterm of t, a contradiction.

(iii) $p = b_K(r)$ and $c = q_K(s)$ for some $r, s \in K$. This case is similar to the preceding one.

(iv) $p = b_K(r)$, $q = c_K(s)$ and $r_K(d) = a = s_K(e)$ for some $r, s \in K$. If $r = x = s$ then we get a contradiction with $t \in M$. Hence either $r \neq x$ or $s \neq x$. However p, q are not reducible and so either $b = x$ or $c = x$. Let $b = x$ (the other case is similar). If $c = x$ then $p = r$, $q = s$ and $p_K(d) = a = q_K(e)$, a contradiction with 5.9 and 4.7. Hence $c \neq x$, consequently $s = x$, $q = c$ and $bc = xq$ is a subterm of t, a contradiction.

5.11. __Lemma.__ $p_K(a), q_K(a) \in M$ for every $a \in M$.

__Proof.__ Only for p. We shall proceed by induction on subterms. Let bc be a subterm of p, $b_K(a), c_K(a) \in M$, $b_K(a) = p_K(d)$ and $c_K(a) = q_K(e)$ for some $d, e \in K$. Since $l(p) \geq l(b)$, $p = b_K(r)$ and $r_K(a) = d$ for some $x \neq r \in K$. Since p is not reducible, $b = x$ and $p = r$. If $q = c_K(s)$ for some $s \in K$ then either $s = x$ and $x.q$ is a subterm of p, a contradiction, or $c = x$ and $q = s$, $p_K(d) = a = q_K(e)$, a contradiction. Thus $c = q_K(m)$ and $bc = x.q_K(m)$ is a subterm of p, a contradiction.

We shall define a new binary operation $\circ$ on the set M. If a, $b \in M$ and $ab \in M$ then we put $a \circ b = ab$. Let a, $b \in M$ and $ab \notin M$. Then there are r, $s \in K$ such that $ab = p_K(r).q_K(s)$. As it is easy to see, $r \in M$ and r, s are determined uniquely. We put $a \circ b = t_K(r)$ (see 5.10). The following lemma is obvious from 4.7.

5.12. Lemma. $aa \in M$ for every $a \in M$.

The next lemma is an easy consequence of 4.7, 5.10, 5.11, 5.12.

5.13. Lemma. (i) $p_{M(\circ)}(a) = p_K(a)$, $q_{M(\circ)}(a) = q_K(a)$ and $t_{M(\circ)}(a) = t_K(a)$ for every $a \in M$.

(ii) $M(\circ)$ is a commutative groupoid without idempotent elements.

(iii) $t_{M(\circ)}(a) = p_{M(\circ)}(a) \circ q_{M(\circ)}(b)$ for all a, $b \in M$.

(iv) $p_{M(\circ)}(a) \neq q_{M(\circ)}(b)$ for all a, $b \in M$.

5.14. Lemma. Let t, p, $q \in K$ be such that $p \neq q$, $l(p) = l(q)$ is a prime and $l(t) \leq l(p)$. Then t, p, q satisfy the conditions (1),...,(7).

Proof. Easy.

5.15. Theorem. Let E (resp. K) be the absolutely free (resp. free commutative) groupoid generated by x, y (resp. x) and $\psi : E \longrightarrow K$ be the homomorphism such that $\psi(x) = x = \psi(y)$. Let t, p, $q \in E$ be such that var $(p) = \{x\} =$ var $(t)$, var $(q) = \{y\}$ and $\psi(t)$, $\psi(p)$, $\psi(q)$ satisfy the conditions (1),...,(7). Then the variety Mod $(xy \hat{=} yx,\ t \hat{=} pq)$ has non-surjective epimorphisms.

Proof. Apply 5.6 and 5.13.

5.16. Corollary. Let t, p, $q \in E$ be such that var(p) = $= \{x\} =$ var(t), var (q) $= \{y\}$, $l(p) = l(q)$ is a prime, $l(t) \leq l(p)$ and $\psi(p) \neq \psi(q)$. Then Mod (xy $\hat{=}$ yx, t $\hat{=}$ pq) has non-surjective epimorphisms.

5.17. Example. The varieties Mod (xy $\hat{=}$ yx, x $\hat{=}$ $\hat{=}$ (x.xx)(y(yy.yy))) and Mod (xy $\hat{=}$ yx, xx.xx $\hat{=}$ $\hat{=}$ ((x.xx)(xx))(y(y(y.yy)))) have non-surjective epimorphisms.

The following lemma is evident.

5.18. Lemma. Let p be a symmetric groupoid term. Then every groupoid from Mod (p(x) $\hat{=}$ p(y)) contains exactly one idempotent.

5.19. Proposition. Let $\xi : E \longrightarrow E$ be the endomorphism such that $\xi(x) = x = \xi(y)$. Let t, p, $q \in E$ be such that var (t) $= \{x\} =$ var (p), var (q) $= \{y\}$, $\xi(p) = \xi(q)$ and t is symmetric. Then the variety $\mathcal{A}$ = Mod (xy $\hat{=}$ yx, t $\hat{=}$ pq) has the strong amalgamation property.

Proof. Let G, $H \in \mathcal{A}$ and $A = G \cap H$ be a subgroupoid of both G and H. Clearly, $\mathcal{A} \subseteq$ Mod (t(x) $\hat{=}$ t(y)), and consequently A contains an idempotent e. Further, $t_A(a) = t_G(b) = t_H(c) = e$ for all $a \in A$, $b \in G$ and $c \in H$. Put $B = G \cup H$ and define ab = e = ba for all $a \in G$, $b \in H$, a, $b \notin A$. It is visible that $B \in \mathcal{A}$

5.20. Example. The variety Mod (xy $\hat{=}$ yx, xx.xx $\hat{=}$ $\hat{=}$ ((xx)(x.xx))((yy)(y.yy))) has the strong amalgamation property, and hence it has only surjective epimorphisms.

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