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Commentationes Mathematicae Universitatis Carolinae, Vol. 18 (1977), No. 2, 351--361

Persistent URL: <http://dml.cz/dmlcz/105779>

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COMPLETION OF SEQUENTIAL CAUCHY SPACES

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Abstract: We study two types of sequential Cauchy spaces projectively generated by classes of functions, their completions, and their mutual relations.

Key words: Sequential Cauchy space, completion, convergence space, sequential envelope.

AMS: 54D55

Ref. Ž.: 3.961.1

1. Introduction. For the reader's convenience we recall in this section some basics about (sequential) Cauchy spaces.

Notation 1.1. If $\langle x_n \rangle, \langle y_n \rangle$ are two sequences, then $\langle x_n \rangle \wedge \langle y_n \rangle$ denotes a sequence $\langle z_n \rangle$ defined as follows: $z_1 = x_1, z_2 = y_1, z_3 = x_2, z_4 = y_2, \dots$, i.e. $x_n = z_{2n-1}, y_n = z_{2n}$.

Definition 1.2. A Cauchy space is a pair (X, L) , where X is a set and L a collection of sequences ranging in X such that

- (1) $\langle x \rangle \in L$ for each $x \in X$;
- (2) $\langle x_n \rangle \in L$ implies $\langle x'_n \rangle \in L$ for each subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$;
- (3) if $\langle x_n \rangle, \langle y_n \rangle \in L$ and there are subsequences $\langle x'_n \rangle$ of $\langle x_n \rangle$ and $\langle y'_n \rangle$ of $\langle y_n \rangle$ such that $x'_n = y'_n, n \in \mathbb{N}$, then $\langle x_n \rangle \wedge \langle y_n \rangle \in L$; and

(4) if $\langle x_n \rangle \wedge \langle x \rangle \in L$ and $\langle x_n \rangle \wedge \langle y \rangle \in L$, then $x = y$.

If (X, L) is a Cauchy space, then L is called a Cauchy structure for X . If L satisfies the additional condition

(5) $\langle x_n \rangle \in L$ whenever

(a) each subsequence $\langle x'_n \rangle$ of $\langle x_n \rangle$ contains a subsequence $\langle x''_n \rangle$ of $\langle x'_n \rangle$ such that $\langle x''_n \rangle \in L$; and

(b) if $\langle x'_n \rangle$ and $\langle x''_n \rangle$ are subsequences of $\langle x_n \rangle$ such that $\langle x'_n \rangle, \langle x''_n \rangle \in L$, then $\langle x'_n \rangle \wedge \langle x''_n \rangle \in L$;

then (X, L) is said to be a *Cauchy space.

The effect of condition (5) can be brought out by considering the real line with its usual metric. Every bounded sequence of real numbers has a Cauchy subsequence. Hence, every bounded sequence of real numbers satisfies condition (a). Yet every bounded sequence of real numbers is not Cauchy in the usual sense because (b) is lacking; e.g. consider the sequence 0, 1, 0, 1, 0, 1,

A Cauchy space (X, L) induces a convergence space $(X, \mathcal{L}, \lambda)$ in the following natural way: $x = \mathcal{L}\text{-lim } x_n$ iff $\langle x_n \rangle \wedge \langle x \rangle \in L$. Moreover, if (X, L) is a *Cauchy space, then $\mathcal{L} = \mathcal{L}^*$. The topological modification λ^{ω_1} of λ will be called a topological closure for X . A subspace Y of X is topologically dense in X if $\lambda^{\omega_1} Y = X$. A Cauchy space is said to be complete if each Cauchy sequence converges in the induced convergence space. A mapping $f: (X_1, L_1) \rightarrow (X_2, L_2)$ is said to be Cauchy-continuous if $\langle x_n \rangle \in L_1$ implies $\langle f(x_n) \rangle \in L_2$. The set of all Cauchy-continuous functions on (X, L) is denoted by $\hat{C}(X, L)$. The set

$M = \{ \langle f_m \rangle \in (\hat{C}(X, L))^N; \lim_{m, n \rightarrow \infty} f_m(x_n) \text{ exists for each } \langle x_n \rangle \in L \}$ is a Cauchy structure for $\hat{C}(X, L)$ and is said to be the continuous Cauchy structure. The space $(\hat{C}(\hat{C}(X, L), M), M)$ is denoted by $(\hat{C}^2(X, L), M^2)$. The evaluation mapping $ev_x: (X, L) \rightarrow (\hat{C}^2(X, L), M^2)$ is defined by $ev_x(x) = \Phi_x$, where for $f \in \hat{C}(X, L)$ we define $\Phi_x(f) = f(x)$; it is always Cauchy-continuous. If it is a Cauchy-embedding (i.e. a Cauchy-homeomorphism into), then (X, L) is said to be \hat{C} -embedded.

2. Projective generations of Cauchy structures.

Proposition and definition 2.1. Let (X, L) be a Cauchy space and $D \subset \hat{C}(X, L)$, D separates points of X . Let $L_D = \{ \langle x_n \rangle \in X^N; \lim f(x_n) \text{ exists whenever } f \in D \}$ and $L_d = \{ \langle x_n \rangle \in X^N; \lim_{m, n \rightarrow \infty} f_m(x_n) \text{ exists whenever } \langle f_m \rangle, f_m \in D \text{ is a Cauchy sequence in } (\hat{C}(X, L), M) \}$. Then L_D and L_d are $*$ Cauchy structures for X and $L \subset L_d \subset L_D$. If $L = L_D$, then L , resp. (X, L) , is said to be projectively generated by D. If $L = L_d$, then L , resp. (X, L) , is said to be c-projectively generated by D.

It follows immediately that if a space is projectively generated by D , then it is also c-projectively generated by D . The converse statement is not true in general as it will be shown by a counterexample (see Proposition 4.7). In

[I - K] it was proved that for $D = C(X, L)$ the following are equivalent:

- (a) (X, L) is \hat{C} -embedded; (b) $L = L_D$; (c) $L = L_d$ (the original notation is $L_D = L_P, L_d = L_M$).

3. d-completion.

Definition 3.1. Let (X,L) be a Cauchy space c-projectively generated by $D \subset \hat{C}(X,L)$. A complete Cauchy space (X_1, L_1) is said to be a d-completion of (X,L) if

(a) (X,L) is a topologically dense subspace of (X_1, L_1) ;

(b) for each $f \in D$ there is $\bar{f} \in \hat{C}(X_1, L_1)$ such that

$$f = \bar{f} \upharpoonright X, \text{ i.e. } D \subset \hat{C}(X_1, L_1) \upharpoonright X;$$

(c) (X_1, L_1) is c-projectively generated by

$$\bar{D} = \{\bar{f} \in \hat{C}(X_1, L_1); \bar{f} \upharpoonright X \in D\}; \text{ and}$$

(d) \bar{D} and D endowed with the corresponding continuous Cauchy structures are Cauchy-homeomorphic under the natural correspondence, i.e. the correspondence $\bar{f} \rightarrow \bar{f} \upharpoonright X = f$ is one-to-one and $\langle \bar{f}_n \rangle, \bar{f}_n \in \bar{D}$, is a Cauchy sequence in $(\hat{C}(X_1, L_1), M)$ iff $\langle f_n \rangle, f_n = \bar{f}_n \upharpoonright X$, is a Cauchy sequence in $(\hat{C}(X,L), M)$.

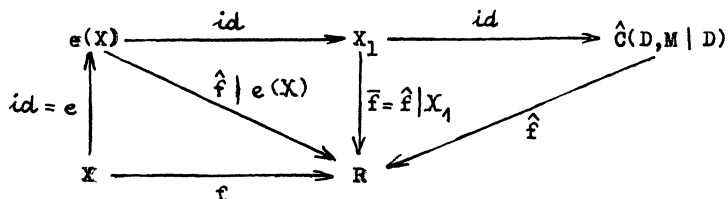
Lemma 3.2. Let (X,L) be a Cauchy space c-projectively generated by $D \subset \hat{C}(X,L)$, $(D, M \upharpoonright D)$ the subspace of $(\hat{C}(X,L), M)$, and e a mapping of (X,L) into $(\hat{C}(D, M \upharpoonright D), M)$ defined as follows: $e(x) = \Phi_x$, where for $f \in D$ we define $\Phi_x(f) = f(x)$. Then e is a Cauchy embedding.

Lemma 3.2 was proved in [I - K] in the special case of $D = \hat{C}(X,L)$. The proof of the general case is similar.

Theorem 3.3. Let (X,L) be a Cauchy space c-projectively generated by $D \subset \hat{C}(X,L)$. Then there exists a d-completion of (X,L) .

Proof. It follows from Lemma 3.2 that identifying x with $e(x)$ we can consider (X,L) as a subspace of $(\hat{C}(D, M \upharpoonright D), M)$. We shall prove that the subspace (X_1, L_1) of $(\hat{C}(D, M \upharpoonright D), M)$, where

X_1 is the topological closure of X in $(\hat{C}(D, M | D), M)$ and $L_1 = M | X_1$, is a d -completion of (X, L) . It was proved in [I - K] that $(\hat{C}(D, M | D), M)$ is a complete space. Thus the closed subspace (X_1, L_1) of $(\hat{C}(D, M | D), M)$ is complete. We are to prove that (X_1, L_1) satisfies conditions (a) - (d) of Definition 3.1. Condition (a) follows from the construction of (X_1, L_1) . It was proved in [F] that the space $(\hat{C}(X, L), M)$ is \hat{C} -embedded. Thus the subspace $(D, M | D)$ is also \hat{C} -embedded, and hence the evaluation mapping $ev_D: (D, M | D) \rightarrow (\hat{C}^2(D, M | D), M^2)$ is a Cauchy embedding. Consequently, for each $f \in D$ the image $ev_D(f) = \hat{f}$ is a Cauchy-continuous function on $(\hat{C}(D, M | D), M)$. Since $\hat{f}(\Phi) = \Phi(f)$ for each $\Phi \in \hat{C}(D, M | D)$, we have $\hat{f}(x) = f(x)$ for each $\Phi_x = x \in X$. Hence $\bar{f} = \hat{f} | X_1$ is a Cauchy-continuous extension of f onto (X_1, L_1) and condition (b) is satisfied. The construction of \bar{f} is shown on the following diagram:



Now, we shall prove condition (d). It follows by a standard topological argument that the extension \bar{f} of f is uniquely determined. Hence the natural correspondence $\bar{f} \rightarrow \bar{f} | X = f$ is one-to-one. Clearly, if $\langle \bar{f}_n \rangle$, $\bar{f}_n | X \in D$, is a Cauchy sequence in $(\hat{C}(X_1, L_1), M)$, then $\langle f_n \rangle$, $f_n = \bar{f}_n | X$, is a Cauchy sequence in $(\hat{C}(X, L), M)$. Conversely, let $\langle f_n \rangle$ be a Cauchy sequence in $(D, M | D)$. Since ev_D is a Cauchy embedding, the se-

quence $\langle \hat{f}_n \rangle$, $\hat{f}_n = \text{ev}_D(f_n)$, is a Cauchy sequence in $(\hat{C}^2(D, M | D), M^2)$. Hence $\langle \bar{f}_n \rangle$, $\bar{f}_n | X = f_n$, is a Cauchy sequence in $(\hat{C}(X_1, L_1), M)$.

It remains to prove condition (c). Let $\langle \Phi_n \rangle$ be a sequence in $X_1 \subset \hat{C}(D, M | D)$ such that

(1) $\lim_{m, n \rightarrow \infty} \bar{f}_m(\Phi_n)$ exists whenever $\langle \bar{f}_m \rangle$, $\bar{f}_m \in \bar{D}$, is a Cauchy sequence in $(\hat{C}(X_1, L_1), M)$.

Since $\bar{f}_m(\Phi_n) = \Phi_n(f_m)$, $f_m = \bar{f}_m | X$, it follows from (d) that (1) is equivalent to

(2) $\lim_{m, n \rightarrow \infty} \Phi_n(f_m)$ exists whenever $\langle f_m \rangle$ is a Cauchy sequence in $(D, M | D)$.

Thus $\langle \Phi_n \rangle \in L_1$ and the proof is complete.

Theorem 3.4. Let (X, L) be a Cauchy space c -projectively generated by $D \subset \hat{C}(X, L)$. If (X_1, L_1) and (X_2, L_2) are two d -completions of (X, L) , then there is a Cauchy homeomorphism $h: (X_1, L_1) \rightarrow (X_2, L_2)$ such that $h(x) = x$ for each $x \in X$.

Proof. For $i = 1, 2$, denote by $D_i = \{f \in \hat{C}(X_i, L_i); f | X \in D\}$, by $(D_i, M | D_i)$ the subspace of $(\hat{C}(X_i, L_i), M)$, and by $(D, M | D)$ the subspace of $(\hat{C}(X, L), M)$. It follows from (d) in Definition 3.2 that $(D_i, M | D_i)$ and $(D, M | D)$ are Cauchy-homeomorphic under the natural correspondence. Consequently, $\varphi: (D_2, M | D_2) \rightarrow (D_1, M | D_1)$, where for $f \in D_2$ its image $\varphi(f)$ is determined by $\varphi(f) | X = f | X$, and hence also its first conjugate $\varphi^*: (\hat{C}(D_1, M | D_1), M) \rightarrow (\hat{C}(D_2, M | D_2), M)$, $\varphi^*(\Phi) = \varphi \circ \Phi$, are Cauchy homeomorphisms. It follows from Lemma 3.2 that identifying x with $e_1(x)$, where for $f \in D_1$ we define $(e_1(x))(f) = f(x)$, we can consider the complete space (X_1, L_1) as a closed subspace of $(\hat{C}(D_1, M | D_1), M)$.

Now, an easy computation shows that for each $x \in X$ we have $\varphi^*(x) = x$.

$$\begin{array}{ccccc}
 X_1 & \xleftarrow{id} & X & \xrightarrow{\quad} & X_2 \\
 id = e \downarrow & & & & \downarrow id = e_2 \\
 C(D_1, M|D_1) & \xrightarrow{\quad} & & \xrightarrow{\varphi^*} & C(D_2, M|D_2)
 \end{array}$$

Since X is topologically dense in (X_1, L_1) , it follows by a standard topological argument that $h = \varphi^*|X_1$ is the desired Cauchy homeomorphism.

4. D-completion.

Definition 4.1. Let (X, L) be a Cauchy space projectively generated by $D \subset \hat{C}(X, L)$. A complete Cauchy space (X_1, L_1) is said to be a D -completion of (X, L) if

- (a) (X, L) is a topologically dense subspace of (X_1, L_1) ;
- (b) for each $f \in D$ there is $\bar{f} \in \hat{C}(X_1, L_1)$ such that $f = \bar{f}|X$, i.e. $D \subset \hat{C}(X_1, L_1)|X$; and
- (c) (X_1, L_1) is projectively generated by $D = \{\bar{f} \in \hat{C}(X_1, L_1); \bar{f}|X \in D\}$.

Proposition 4.2. Let (X, L) be a Cauchy space projectively generated by $D \subset \hat{C}(X, L)$ and $(X, \mathcal{L}^*, \lambda)$ the associated convergence space. Then:

- (a) $D \subset C(X)$ and $(X, \mathcal{L}^*, \lambda)$ is D -sequentially regular.
- (b) L is the set of all D -fundamental sequences in $(X, \mathcal{L}^*, \lambda)$.
- (c) $(X, \mathcal{L}^*, \lambda)$ is D -sequentially complete iff (X, L)

is complete.

The straightforward proof is omitted.

Proposition 4.3. Let $(X, \mathcal{L}^*, \lambda)$ be a D-sequentially regular convergence space and L the set of all D-fundamental sequences. Then:

- (a) L is a $*$ Cauchy structure for X.
- (b) $D \subset \hat{C}(X, L)$ and (X, L) is projectively generated by D.
- (c) $(X, \mathcal{L}^*, \lambda)$ is associated with (X, L) .
- (d) (X, L) is complete iff $(X, \mathcal{L}^*, \lambda)$ is D-sequentially complete.

The straightforward proof is omitted.

Theorem 4.4. Let (X, L) be a Cauchy space projectively generated by $D \subset \hat{C}(X, L)$. Then there exists a D-completion of (X, L) .

Proof. Let $(X, \mathcal{L}, \lambda)$ be the convergence space associated with (X, L) . It follows from (a) in Proposition 4.2 that $(X, \mathcal{L}, \lambda)$ is D-sequentially regular. Let $(X_1, \mathcal{L}_1, \lambda_1)$ be a D-sequential envelope of $(X, \mathcal{L}, \lambda)$, $\bar{D} = \{\bar{f} \in C(X_1); \bar{f} \upharpoonright X \in D\}$, and L_1 the set of all \bar{D} -fundamental sequences in X_1 . It follows from Proposition 4.2 and Proposition 4.3 that (X_1, L_1) is a D-completion of (X, L) .

Note 4.5. Let $(X, \mathcal{L}^*, \lambda)$ be a D-sequentially regular convergence space. Let L be the set of all D-fundamental sequences in X. It follows from Proposition 4.3 that (X, L) is a $*$ Cauchy space projectively generated by $D \subset \hat{C}(X, L)$. Let (X_1, L_1) be a D-completion of (X, L) . Using Proposition 4.2 and Proposition 4.3 it is easy to see that the convergence space $(X_1, \mathcal{L}_1, \lambda_1)$ associated with (X_1, L_1) is a D-sequential enve-

lope of $(X, \mathcal{C}^*, \lambda)$.

Theorem 4.6. Let (X, L) be a Cauchy space projectively generated by $D \subset \hat{C}(X, L)$. If (X_1, L_1) and (X_2, L_2) are two D-completions of (X, L) , then there is a Cauchy homeomorphism $h: (X_1, L_1) \rightarrow (X_2, L_2)$ such that for each $x \in X$ we have $h(x) = x$.

Proof. Let $(X, \mathcal{C}, \lambda)$ be the convergence space associated with (X, L) and $(X_i, \mathcal{C}_i, \lambda_i)$ the convergence space associated with (X_i, L_i) , $i = 1, 2$. It follows from Note 4.5 that $(X_i, \mathcal{C}_i, \lambda_i)$ is a D-sequential envelope of $(X, \mathcal{C}, \lambda)$. Hence there is a homeomorphism $h: (X_1, \mathcal{C}_1, \lambda_1) \rightarrow (X_2, \mathcal{C}_2, \lambda_2)$ such that for each $x \in X$ we have $h(x) = x$ (cf. Theorem 5 in [N]). Since (X_i, L_i) are complete space, $h: (X_1, L_1) \rightarrow (X_2, L_2)$ is a Cauchy homeomorphism.

5. Example.

Definition 5.1. Let $X \neq \emptyset$ and $\langle x_n \rangle, \langle y_n \rangle \in X^{\mathbb{N}}$. We say that $\langle y_n \rangle$ is derived from $\langle x_n \rangle$, in symbols $\langle y_n \rangle \rightarrow \langle x_n \rangle$, if $F(\langle y_n \rangle) \supset F(\langle x_n \rangle)$, where $F(\langle x_n \rangle)$ denotes the filter generated by sections of a sequence $\langle x_n \rangle$.

Example 5.2. Let $X_2 = (\bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} (x_{mn})) \cup (\bigcup_{m \in \mathbb{N}} (x_m)) \cup (x_0)$. Let $\alpha \in \mathbb{N}^{\mathbb{N}}$, $m_0 \in \mathbb{N}$, $A \subset (\bigcup_{m \in \mathbb{N}} (x_{m\alpha(m)}))$, and $f \in \{0, 1\}^{X_2}$ a function on X_2 defined as follows:

$$f(x) = 1 \text{ for } x \in (A \cup (\bigcup_{n \in \mathbb{N}} (x_{m_0 n})) \cup (x_{m_0})),$$

$$f(x) = 0 \text{ otherwise.}$$

Let \bar{D} be the set of all such functions and (X_2, L_2) the Cauchy space projectively generated by \bar{D} . Let $X = \bigcup_{m \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} (x_{mn})$,

$X_1 = X \cup \left(\bigcup_{m \in \mathbb{N}} \langle x_m \rangle \right)$, $L = L_2 | X$, $L_1 = \{ \langle z_n \rangle \in X_1^{\mathbb{N}}; \langle z_n \rangle \rightarrow \langle x \rangle, x \in X_1, \text{ or } \langle z_n \rangle \rightarrow \langle \langle x_{mn} \rangle \wedge \langle x_m \rangle \rangle, m \in \mathbb{N} \}$, and $D = \bar{D} | X$.

Since (X, L) is clearly projectively generated by D it is also c -projectively generated by D and hence (X, L) possesses both a D -completion and a d -completion.

Proposition 5.3. For $\hat{D} = \bar{D} | X_1$ the space (X_1, L_1) is c -projectively generated by \hat{D} , but not projectively generated by \hat{D} .

Hint. $L_2 = \{ \langle z_n \rangle \in X_2^{\mathbb{N}}; \langle z_n \rangle \rightarrow \langle x \rangle, x \in X_2, \text{ or } \langle z_n \rangle \rightarrow \langle \langle x_{mn} \rangle \wedge \langle x_m \rangle \rangle, \text{ or } \langle z_n \rangle \rightarrow \langle \langle x_m \rangle \wedge \langle x_0 \rangle \rangle \}$ and $\langle x_m \rangle \in (L_2 | X_1 - L_1)$.

Proposition 5.4. (X_1, L_1) is a d -completion of (X, L) .

Hint. $L = \{ \langle z_n \rangle \in X^{\mathbb{N}}; \langle z_n \rangle \rightarrow \langle x \rangle, x \in X, \text{ or } \langle z_n \rangle \rightarrow \langle x_{mn} \rangle, m \in \mathbb{N} \}$.

Proposition 5.5. (X_2, L_2) is a D -completion of (X, L) .

Proposition 5.6. \bar{D} and D endowed with the corresponding continuous Cauchy structures are not Cauchy-homeomorphic under the natural correspondence:

Proof. For otherwise (X_2, L_2) would be also a d -completion of (X, L) , which would imply the existence of a Cauchy homeomorphism $h: (X_1, L_1) \rightarrow (X_2, L_2)$ such that for each $x \in X$ we have $h(x) = x$.

Note 5.7. This shows that the condition (d) in Definition 3.1 is necessary and sufficient for the uniqueness of the d -completion up to a commuting Cauchy homeomorphism (cf. Theorem 3.4).

Note 5.8. Let (X, L) be a \hat{C} -embedded Cauchy space. Since for $D = \hat{C}(X, L)$ we have $L = L_D = L_D$, it follows immediately that a d -completion of (X, L) is also a D -completion of (X, L) . Consequently, the two completions are equivalent. It might be of some interest to characterize classes $D \subset \hat{C}$ for which the two completions are equivalent.

R e f e r e n c e s

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(Obitum 16.12. 1976)