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PRODUCTIVE REPRESENTATIONS OF SEMIGROUPS BY PAIRS OF STRUCTURES

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Abstract: We prove that for any commutative semigroup \((S,+)\) there exists a collection \(\{r(s) \mid s \in S\}\) of complete metric spaces such that for every \(s_1, s_2 \in S\),

1. \(r(s_1 + s_2)\) is isometric to \(r(s_1) \times r(s_2)\) and
2. if \(s_1 \neq s_2\) then \(r(s_1)\) is not homeomorphic to \(r(s_2)\).

Key words: Semigroup, representation, product, metric space, box-product.

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1. Let us begin with a definition.

Definition. Let \(K, H\) be categories, \(K\) have finite products. Let \(\mathcal{F} : K \to H\) be a functor. Let \((S,+)\) be a commutative semigroup. Any mapping

\[ r: S \to \text{obj} K \]

is called an \(\mathcal{F}\)-productive representation of \((S,+)\) if

1. for any \(s_1, s_2 \in S\), \(r(s_1 + s_2)\) is isomorphic to \(r(s_1) \times r(s_2)\) in \(K\);
2. if \(s_1, s_2 \in S\), \(s_1 \neq s_2\), then \(\mathcal{F}(r(s_1))\) is not isomorphic to \(\mathcal{F}(r(s_2))\) in \(H\).

In \([T_2]\), a representation of \((S,+)\) by products in a category
K is introduced. It is a special case of the above definition with \( K = H \) and \( \mathcal{F} = \text{ident} \). The dual definitions of \( \mathcal{F} \)-coproductive representation is evident.

2. Some of the known results give \( \mathcal{F} \)-productive representations of some semigroups. Let us recall some of them.

A) Let \( \mathcal{L} \) be the category of lattices and all lattice-homomorphisms; let \( \mathcal{L} \mathcal{L} \) be the category of all linear lattices and all linear lattice-homomorphisms. Let \( \mathcal{L} : \mathcal{L} \mathcal{L} \to \mathcal{L} \) be the functor which assigns to each linear lattice its underlying lattice. Then

any Abelian group and any countable commutative semigroup have \( \mathcal{L} \)-productive representations.

B) Let \( \mathcal{R} \) be the category of all commutative rings with unit (and all their unit-preserving homomorphisms), let \( \mathcal{S} \) be the category of all commutative semigroups with unit. Let \( \mathcal{R} : \mathcal{R} \to \mathcal{S} \) be the functor which assigns to each ring its multiplicative semigroup. Then

any Abelian group and any countable commutative semigroup have \( \mathcal{R} \)-productive representations.

C) Let \( \mathcal{B} \) be the category of all Banach spaces and all bounded linear operators with the norm \( \leq 1 \), let \( \mathcal{B} \mathcal{A} \) be the category of all Banach algebras. Let \( \mathcal{B} : \mathcal{B} \mathcal{A} \to \mathcal{B} \) be the functor which assigns to each Banach algebra its underlying Banach space. Then

any Abelian group and any countable commutative semigroup have \( \mathcal{B} \)-productive representations.

In all these cases, the \( \mathcal{L} \)- or \( \mathcal{R} \)- or \( \mathcal{B} \)-productive representations are obtained as follows. By [AKT], any Abelian
group has a representation by coproducts of Boolean spaces (i.e. compact Hausdorff zero-dimensional spaces), in other words, for any Abelian group $G$ there exists a collection \( \{ r(g) \mid g \in G \} \) of pairwise non-homeomorphic Boolean spaces such that $r(g_1 \cdot g_2)$ is always homeomorphic to the coproduct $r(g_1) \sqcup r(g_2)$ of $r(g_1)$ and $r(g_2)$. The analogous result for all countable commutative semigroups is proved in [K] (here, $r(g)$ are metrizable).

Consider the sets $C(r(g))$ of all real-valued continuous functions on these spaces $r(g)$. They can be structured in a lot of ways: As linear lattices and lattices for $A$, as rings and semigroups for $B$, as Banach algebras and Banach spaces for $C$). Structured as a linear lattice or ring or Banach algebra, $C(r(g_1) \sqcup r(g_2))$ is isomorphic to $C(r(g_1)) \times C(r(g_2))$ in the corresponding category. Since $r(g_1)$ is not homeomorphic to $r(g_2)$, $C(r(g_1))$ is not isomorphic to $C(r(g_2))$, structured as lattices (by the Birkhoff-Kaplansky theorem) or Banach spaces (by the Banach-Stone theorem) or multiplicative semigroups (by Milgram [M]).

Let us notice that if $\mathcal{F}: K \to \mathbb{H}$ preserves finite products and a semigroup has an $\mathcal{F}$-productive representation, then it has a representation by products in $\mathbb{H}$ in the sense of $[T_2]$. Hence, if a functor $\mathcal{F}$ from an arbitrary category into the category Set of all sets or into the category Lin of all linear spaces preserves finite products, then no non-trivial Abelian group has an $\mathcal{F}$-productive representation.

3. Let $CM$ be the category of all complete metric spaces with diameter $\leq 1$ and all their contractions (we re-
call that a mapping \( c \) is a contraction if \( \text{dist}(c(x), c(y)) \leq \epsilon \text{dist}(x, y) \) for all \( x, y \). Let us notice that isomorphisms in \( \mathcal{CM} \) coincide with isometries and a product-metric \( d \) of \( d_1 \) and \( d_2 \) is given by the usual formula
\[
d((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.
\]
Let \( \text{Top} \) be the category of all topological spaces and all their continuous mappings. Here isomorphisms coincide with homeomorphisms. Let
\[\mathcal{M} : \mathcal{CM} \rightarrow \text{Top}\]
be the functor which assigns to each metric space its underlying topological space. The aim of this note is to prove the following theorem:

**Theorem.** Every commutative semigroup has an \( \mathcal{M} \)-productive representation.

Every commutative semigroup has a representation by products of uniform, proximity and topological spaces; by [AK], every \( C \)-embeddable semigroup has a representation by products of metrizable topological spaces. The above theorem strengthens all these results.

4. First, we sketch modifications of the general method, described in [\( T_2 \)]. If a semigroup \( S \) has an \( \mathcal{F} \)-productive representation, then any of its subsemigroups has also an \( \mathcal{F} \)-productive representation. Consequently, it is sufficient to investigate \( \mathcal{F} \)-productive representation of "universal semigroups" (this means universal for some class of semigroups with respect to an embedding of semigroups). Denote by \( N \) the additive semigroup of all non-negative inte-
gers, by $\mathbb{N}^m$ its $m$-th power (with the operation given pointwise) and by $\exp \mathbb{N}^m$ the semigroup of all its subsets (with the operation given by $A \ast B = \{ a \ast b \mid a \in A, b \in B \}$).

By $[T_3]$, any commutative semigroup $S$ can be embedded in $\exp \mathbb{N}^m$ with $\mathbb{N} = \mathbb{N}_0^\ast \text{card } S$. Hence, we shall investigate $\mathcal{F}'$-productive representations of the semigroups $\exp \mathbb{N}^m$.

5. We shall use the following notation and conventions. Isomorphism in a category will be denoted by $\simeq$, product by $\prod$ (or $\times$ for finite collections), coproduct by $\amalg$. The product of the empty collection is a terminal object (it can be added to a category whenever it is missing).

If $a$ is an arbitrary object of a category with finite products, then $a^0$ is the terminal object, $a^1 \simeq a$, $a^{n+1} \simeq a \times a^n$.

We say that a category $K$ with all products and coproducts is distributive (see $[T_2]$) if

$$\left( \prod_{i \in I} a_i \right) \times \left( \amalg_{j \in J} b_j \right) \simeq \left( \amalg_{i \in I, j \in J} a_i \times b_j \right).$$

We say that an object $a$ is a summand of $b$ if $b \simeq a + c$ for an object $c$.

6. Let $K$ be a category with products, let $\mathcal{F} : K \rightarrow \mathcal{H}$ be a functor. Let $\mathcal{Z}$ be a set of objects of $K$.

For any $f \in \mathbb{N}^\mathcal{Z}$, denote by $\mathcal{Z}(f)$ the product $\prod_{z \in \mathcal{Z}} f(z)$ (if $f(z) = 0$, $\mathcal{Z}(f)$ is the terminal object). We say that $\mathcal{Z}$ is an $\mathcal{F}$-independent set of objects of $K$ if for every $f \in \mathbb{N}^\mathcal{Z}$, $\mathcal{Z}(f)$.
Whenever \( f \in A \) whenever \( f(\mathcal{L}(f)) \) is a summand of \( f((g_s)_{s \in S}) \), where \( S \) is a set and \( \mathcal{L}(g)_s \approx \mathcal{L}(g) \) for all \( s \in S \).

(This generalizes the notion of productively independent set of objects, see [T,2] and [AK].)

7. Proposition. Let \( K \) be a distributive category, let \( \mathcal{F} : K \to H \) preserve coproducts. Let there exist an \( \mathcal{F} \)-independent set \( \mathcal{Z} \) of objects of \( K \). Then the semigroup \( \exp N^\mathcal{Z} \) has an \( \mathcal{F} \)-productive representation.

Proof. For any \( f \in N^\mathcal{Z} \) denote \( \mathcal{L}(f) = \biguplus_{z \in \mathcal{Z}} \mathcal{L}(z) \); let \( C(f) \) be a coproduct of \( \mathcal{L}^\mu \) copies of \( \mathcal{L}(f) \) with \( \mu = \text{card } \mathcal{Z} \). For \( A \in N^\mathcal{Z} \) put

\[
    r(A) = \biguplus_{f \in A} C(f).
\]

Then \( r \) is an \( \mathcal{F} \)-productive representation of \( \exp N^\mathcal{Z} \). For, if \( A, B \in N^\mathcal{Z} \), then \( r(A + B) \simeq r(A) \times r(B) \) (implied by \( \mathcal{L}(f + g) \simeq \mathcal{L}(f) \times \mathcal{L}(g) \)). If \( A \parallel B \), say if \( A \parallel B \parallel B \), then, for \( f \in A \parallel B \), \( \mathcal{F} \mathcal{L}(f) \) is a summand of \( \mathcal{F} r(A) \) while it cannot be a summand of \( \mathcal{F} r(B) \) because \( \mathcal{Z} \) is \( \mathcal{F} \)-independent. Hence, \( \mathcal{F} r(A) \) is not isomorphic to \( \mathcal{F} r(B) \) in \( H \).

Corollary. Let \( K \) be a distributive category, let \( \mathcal{F} : K \to H \) be a coproduct-preserving functor. Let \( K \) have an arbitrarily large \( \mathcal{F} \)-independent set of objects. Then any commutative semigroup has an \( \mathcal{F} \)-productive representation.

8. Let us examine the category \( \mathcal{CM} \). It has all coproducts (for, if \( \{(x_i,d_i) \mid i \in I\} \) is a collection of ob-
jects, $X_i$ disjoint, put $X = \biguplus_i X_i$, $d(x,y) = d_i(x,y)$ whenever $x, y \in X_i$ for some $i$, $d(x,y) = 1$ otherwise; $d$ is complete whenever all the $d_i$'s are complete; $(X,d)$ is a coproduct in $(CM)$. It has all products (for, if $\{(X_i,d_i)\mid i \in I\}$ is a collection of objects, put $X = \prod_{i \in I} X_i$, $d(x,y) = \sup_{i \in I} d_i(x_i,y_i)$. Clearly, $CM$ is distributive. The functor $M: CM \to Top$ preserves coproducts and finite products, but it does not preserve products in general. To prove the theorem, we have to show that $CM$ contains arbitrarily large sets of $M$-independent sets of objects.

9. If $\{Y_i\mid i \in I\}$ is a collection of topological spaces, denote by $\prod_i Y_i$ their box-product. We recall that a set $\mathcal{Y}$ of topological spaces is called stiff if for any $Y_1, Y_2 \in \mathcal{Y}$ and any continuous mapping $m: Y_1 \to Y_2$ either $m$ is constant or $Y_1 = Y_2$ and $m = \text{ident}$.

Now, let $\mathcal{Y}$ be a set of topological spaces. For any $f \in \mathcal{Y}$ denote by $B(f)$ a topological space with the same underlying set as $\prod_{Y \in \mathcal{Y}} f(Y)$ and such that both the identical mappings

$\prod_{Y \in \mathcal{Y}} f(Y) \to B(f) \to \prod_{Y \in \mathcal{Y}} f(Y)$

are continuous (where $\prod$ denotes product in $Top$).

In the following proposition, $S$ is a set and, for each $g \in S$, $(B(g))_g$ is homeomorphic to $B(g)$. If $x \in B(g)$, $x_g$ means the corresponding subspace of $(B(g))_g$.

**Proposition.** Let $\mathcal{Y}$ be a stiff set of connected Hausdorff spaces. Let $f \in \mathcal{Y}$, $A \subseteq \mathcal{Y}$ be given. If $B(f)$ is homeomorphic to a closed-and-open subset of $\prod_{g \in A \times S} (B(g))_g$, 

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then \( f \in A \).

**Proof.** a) First, let us notice that for any \( Y \in \mathcal{Y} \) and any \( m_1, m_2 \in \mathbb{N} \), the existence of a homeomorphism of \( Y^{m_1} \) into \( Y^{m_2} \) implies \( m_1 \leq m_2 \) (see [H]).

b) For any \( g \in \mathcal{Y} \) and any \( x \in B(g) \), denote by \( B_x(g) \) the subspace of \( B(g) \) consisting of all these points \( y \) which differ from \( x \) only in finitely many coordinates. Clearly, every \( B_x(g) \) is connected. One can see easily by a), that if, for some \( g_1, g_2 \in \mathcal{Y} \) and some \( x \in B(g_1) \), there exists a homeomorphism of \( B_x(g_1) \) into \( B(g_2) \), then \( g_1 \leq g_2 \).

c) Now, let \( h: B(f) \rightarrow \bigcup_{(g, g_0) \in A \times S} (B(g))_{g_0} \) be a homeomorphism onto a closed-and-open subset. Choose \( x \in B(f) \). Since \( B_x(f) \) is connected, there exists \( (g_0, s_0) \in A \times S \) such that \( h(B_x(f)) \subseteq (B(g_0))_{g_0} \). By b), \( f \leq g_0 \). Put \( y = h(x) \). Since \( h(B(f)) \) is a closed-and-open set containing \( y \) and \( (B_y(g_0))_{g_0} \) is connected, it is contained in \( h(B(f)) \). Hence, \( h^{-1} \) defines a homeomorphism of \( B_y(g_0) \) into \( B(f) \). By b), \( g_0 \leq f \). We conclude that \( f = g_0 \in A \).

10. By \([T_1]\), there exist arbitrarily large stiff sets \( \mathcal{Y} \) of connected topological spaces such that any \( Y \in \mathcal{Y} \) can be metrized by a complete metric, say \( d_Y \). We may suppose \( d_Y \leq 1 \). By the previous proposition, \( \mathcal{X} = \{ (Y, d_Y) \mid Y \in \mathcal{Y} \} \) is an \( \mathcal{M} \)-independent set of objects of \( C \mathcal{M} \). This completes the proof of the theorem.

**References**


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