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EVERY FINITE LATTICE CAN BE EMBEDDED IN THE LATTICE OF ALL EQUIVALENCES OVER A FINITE SET

(Preliminary communication)

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Abstract: The theorem given in the title answers in the affirmative a question raised in Ph.M. Whitman [2]. The proof of the theorem is based on graph-theoretical and combinatorial techniques.

Key words: Finite lattice, equivalence lattice, regraph power.

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Here we present a sketch of proof of the theorem in the title. It was first conjectured by Whitman in [2].

Throughout the paper all structures are finite.

Let L, K be two lattices. A mapping $\varphi : L \to K$ is called join-homomorphism, if $\varphi(x \vee y) = \varphi(x) \vee \varphi(y)$ and meet-homomorphism, if $\varphi(x \wedge y) = \varphi(x) \wedge \varphi(y)$ for all $x,y \in L$.

A lattice L is called embeddable, if there exists an embedding (that is an injective join and meet-homomorphism) $\varphi : L \to E_q(A)$ of L in the lattice of all equivalence over a set A.

The least element of L is denoted by 0_L.

Let L be a lattice, $u,v \in L$. If $u \in v$, then we define a

set $L_{u,v} = \{x \in L, v \leq x \text{ or } u \nleq x\}$ and a mapping $\sigma_{u,v}: L \to$
$\to L_{u,v}$

$\sigma_{u,v}(x) = x \vee v$ if $u \leq x$,

$\sigma_{u,v}(x) = x$ if $u \nleq x$.

Lemma 1:

a) $L_{u,v}$ with the ordering induced by L is a lattice

b) $\sigma_{u,v}$ is a surjective join-homomorphism

c) every join-homomorphism $\varphi: L \to K$ such that $\varphi(u) = \varphi(v)$, can be decomposed in the $\sigma_{u,v}: L \to L_{u,v}$ and a join-homomorphism $\psi: L_{u,v} \to K$.

The following theorem uses the fact that for every lattice L there exist a Boolean lattice B and a surjective join-homomorphism $\sigma: B \to L$.

Theorem 1: Let $\mathcal{L}$ be a class of lattices closed under isomorphisms and

1) every Boolean lattice belongs to $\mathcal{L}$

2) $L_{u,v} \in \mathcal{L}$ whenever $L \in \mathcal{L}$, $u,v \in L$ and $u < v$

Then $\mathcal{L}$ is the class of all lattices.

It is known that every Boolean lattice is embeddable. By Theorem 1 it remains only to investigate the operation $L \mapsto L_{u,v}$ in the class of embeddable lattices. To this end the following lemma is a useful tool.

Lemma 2: Let L, K be lattices, $u,v \in L$, $u < v$ and φ: $: L \to K$ a mapping with properties

1) $\varphi: L \to K$ is a join-homomorphism

2) the restriction of φ to $L_{u,v}$, $\varphi_{u,v}: L_{u,v} \to K$ is an injective meet-homomorphism

3) $\varphi(u) = \varphi(v)$

Then $\varphi_{u,v}: L_{u,v} \longrightarrow K$ is a lattice embedding.

To find a mapping $\varphi: L \longrightarrow E_q(A)$ with the properties 1. - 3. several types of constructions are used.

1. Group construction. Let B be the set of all permutations of A. We define a mapping $\varphi: E_q(A) \longrightarrow E_q(B)$ by $(\pi, \rho) \in \varphi(x)$ iff $(\pi \rho^{-1}(a), a) \in x$ for all $a \in A$. Then φ is a lattice embedding. This construction was known already to Birkhoff [1].

2. Regraph construction. By a regraph valued by A we mean a triple $\mathbb{G} = (G, h, \sigma)$, where G is a non-empty set, R is a symmetric antireflexive relation and $\sigma: R \longrightarrow A$ is a mapping. For a given mapping $\varphi: L \longrightarrow E_q(A)$ we define a new mapping $\psi: L \longrightarrow E_q(A \times G)$ called $\mathbb{G}$-power of φ as follows: $\psi(x)$ is the least equivalence containing the relations

$S = \{[(\sigma(g,h),g), (\sigma(h,g),h)] ; (g,h) \in R\}$ and

$S_x = \{[(a,g),(h,g)] ; g \in G \text{ and } (a,b) \in \varphi(x)\}$

If φ is a join-homomorphism, then any its $\mathbb{G}$-power is a join-homomorphis, too. Under certain conditions on the couple $\mathbb{G}$, φ we can prove that the $\mathbb{G}$-power of an injective meet-homomorphism φ is also an injective meet-homomorphism.

3. Perfect regraph construction. A regraph $\mathbb{G} = (G, h, \sigma)$ valued by A is called symmetric, if the valuation $\sigma: R \rightarrow A$ is symmetric, i.e. $\sigma(g,h) = \sigma(h,g)$ for every $(g,h) \in R$. In a symmetric regraph (G,h,σ) an R-chain $g = g_0, \dots, g_k = h$ is called σ-shortest path, if $\{\sigma(g_0,g_1), \sigma(g_1,g_2), \dots, \sigma(g_{k-1},g_k)\} \subseteq$ $\subseteq \{\sigma(h_0,h_1), \dots, \sigma(h_{k-1},h_k)\}$ for every R-chain $g = h_0, h_1, \dots$ $\dots, h_n = h$.

A regraph (G,h,σ) is called perfect, if it is symmetric and for every pair of distinct elements $g,h \in G$ there exists an σ-shortest path $g = g_0,\dots,g_k = h$. If $\mathbb{G} = (G,h,\sigma)$ is a perfect regraph, then the $\mathbb{G}$-power of every embedding φ : $L \longrightarrow E_q(A)$ is an embedding, too.

Example. Cyclic two-valued regraph consists of a cycle of an even length ≥ 4 and a symmetric two-valued valuation σ, which assigns different values to any two incident edges. (We consider $(g,h) \in R$ and $(h,g) \in R$ being a single unoriented edge.) Cyclic two-valued regraphs are perfect.

4. Product of regraphs. If $\mathbb{G}_i = (G_i,h_i,\sigma_i)$ are regraphs valued by A_i for $i = 1,\dots,n$, then $\mathbb{G} = (G,h,\sigma)$ is a product of $\mathbb{G}_i$'s if

1) $G = G_1 \times G_2 \times \dots \times G_n$

2) $[(g_1,\dots,g_n),(h_1,\dots,h_n)] \in R$ iff there exists $j \in \{1,\dots,k\}$ such that $(g_j,h_j) \in R_j$ and $g_i = h_i$ for all $i \neq j$.

3) In this case $\sigma[(g_1,\dots,g_n),(h_1,\dots,h_n)] = \sigma_j(g_j,h_j)$

The fact that product of perfect regraphs is perfect, is easy but of great importance.

Using constructions just listed we can construct new embeddings of a given embeddable lattice, satisfying some special conditions.

Lemma 3: If L is an embeddable lattice, $u \in L$, $u \neq 0_L$, then there exist an embedding $\varphi : L \longrightarrow E_q(A)$ and a set $V \subseteq A$ with properties:

1) for every $a \in A$ there are two different $x,y \in V$ such that $(x,a) \in \varphi(u)$ and $(y,a) \in \varphi(u)$.

2) For every two different $x,y \in V$, if $(x,y) \in \varphi(v)$ then

$v \geq u$.
To find an embedding with the property 1., the group construction can be used only. In the case of 2. a product of cyclic two-valued regraphs and a non trivial combinatorial lemma are needed.

The proof is finished by

Theorem 2: If L is embeddable, then so is $L_{u,v}$ for $u, v \in L$, $u < v$. In the case $u = 0_L$ $L_{u,v}$ is trivially embeddable being a sublattice of an embeddable lattice. So we can assume $0_L \neq u$. Now we take an embedding $\varphi : L \rightarrow E_q(A)$ given by Lemma 3. Further, we construct a new embedding $\psi : L \rightarrow E_q(A \times G)$ - the $\mathbb{G}$ -power of φ, where $\mathbb{G} = (G, h, \sigma)$ is a product of cyclic two-valued regraphs. Then the valuation σ is slightly changed to σ^* so that $\psi^* : L \rightarrow E_q(A \times G)$ - the (G, R, σ^*) - power of φ - identifies u and v. In this step the old assertion about the existence of an Euler cycle in an unoriented graph is used. Finally, we prove that the restriction $\psi^*_{u,v}$: : $L_{u,v} \rightarrow E_q(A \times G)$ of ψ^* is an injective meet-homomorphism. Here we need the theory of non-perfect regraphs. Since the mapping ψ^* is a join-homomorphism being a regraph-power of join-homomorphism φ, it satisfies all assumptions of Lemma 2 and so $\psi^*_{u,v}$ is an embedding of $L_{u,v}$ in $E_q(A \times G)$.

The result was obtained at the end of 1976. The complete proof was submitted for publication to Algebra Universalis.

R e f e r e n c e s

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