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THE SORGENFREY LINE HAS NO CONNECTED COMPACTIFICATION

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Abstract: We answer the question raised by Eric van Douwen during the Conference at Stefanová, February 1977, whether there exists a connected compactification of the Sorgenfrey line. We prove that there is no regular Hausdorff connected space containing the Sorgenfrey line as a dense subspace. We give an example of Hausdorff connected space containing the Sorgenfrey line as a dense subspace.

Key words: Connected space, Sorgenfrey line.

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1. The statements of results. Let $S$ be the Sorgenfrey line, i.e. the set $\mathbb{R}$ of reals with the topology generated by half-open intervals $[x,y)$ of $\mathbb{R}$. If $S$ is a subset of $Y$, then let $U(x,y)$ be the greatest open subset of $Y$ such that $U(x,y) \cap S = [x,y)$.

Theorem. There exists no regular space $Y$ such that $S$ is a dense subspace of $Y$ and

\[(*) \quad (x,y) \cap S - (x,y) \neq \emptyset \text{ for each } (x,y) \in S.

Remark. If $Y$ is a connected space, or if $Y - S$ is connected and $Y$ is compact, then the condition $(*)$ holds.

Corollary 1. There exists no regular connected Hausdorff space containing $S$ as a dense subset.
Corollary 2. There exists no compactification of $S$ with a connected remainder.

2. The proofs. We begin from a

**Lemma.** Let $Y$ be a Hausdorff space containing $S$ as a dense subset. If $p \in \overline{\{x,y\}} \cap (Y - S)$, then there exists a $q$ in $S$ such that $x \leq q \leq y$ and such that for each open neighbourhood $W$ of $p$ the open interval $(q - \varepsilon, q)$ intersects $W$ for each $\varepsilon > 0$.

**Proof of Lemma:** Let $q = \sup \{r \in S : \text{there exists an open neighbourhood } W \text{ of } p \text{ such that } W \cap \{x, r\} = \emptyset \}$. Since $Y$ is Hausdorff, there is an open neighbourhood $W$ of $p$ and there is a point $r$ in $S$ such that $\{x, r\} \cap W = \emptyset$. Therefore $q > x$. If $q > y$, then there are $r > y$ and an open neighbourhood $W$ of $p$ such that $\{x, r\} \cap W = \emptyset$. This contradicts the fact that $p \in \overline{\{x, y\}}$. Hence $q \leq y$. It remains to show that $W \cap (\{q - \varepsilon, q\} = \emptyset$ for each $\varepsilon > 0$ and for each open neighbourhood $W$ of $p$. Suppose not. Then there are $\varepsilon > 0$ and an open neighbourhood $W_1$ of $p$ such that $W_1 \cap (\{q - \varepsilon, q\} = \emptyset$. From the definition of point $q$ there is an open neighbourhood $W_2$ of $p$ such that $W_2 \cap \{x, q - \frac{\varepsilon}{2}\} = \emptyset$. Since $Y$ is Hausdorff, there is an open neighbourhood $W_3$ of $p$ such that $W_3 \cap \{q, q + \varepsilon_1\} = \emptyset$ for any $\varepsilon_1 > 0$. Hence $W \cap \{x, q + \varepsilon_1\} = \emptyset$ where $W = W_1 \cap W_2 \cap W_3$. This contradicts the definition of point $q$.

**Proof of the Theorem.** Suppose that there is a regular Hausdorff space $Y$ containing $S$ as a dense subset and the condition $(\ast)$ holds. We first show that $(\ast \ast)$ for each $x, y$ in $S$ there exist $p$ in $Y - S$ and $p_1, p_2$ in $(x, y)$ such that $p_1 \neq p_2$ and $W \cap (p_1 - \varepsilon, p_2) = \emptyset$ and
$\forall (p_2 - \varepsilon, p_2) \neq \emptyset$ for each $\varepsilon > 0$ and open neighbourhood $W$ of $p$.

Since $Y$ is regular, there is a point $z$ in $(x,y)$ such that $[x,z] \subset U(x,y)$. From the condition (*) there is a point $p$ in $[x,z] \cap S - [x,z)$. Then $p \in [x,z] \subset U(x,y)$ and $p \in S - [x,z)$, and therefore for each open neighbourhood $W$ of $p$ we have $\emptyset \neq W \cap U(x,y) \cap (S - [x,z)) = W \cap U(x,y) \cap (S - [x,z)) = W \cap (z, y)$. This implies that $p \in (z, y)$. Hence there exists a point $p$ in $Y - S$ belonging to $[x,z)$ and $[z, y)$. By the Lemma, there exist $p_1$ and $p_2$ in $S$ such that (*) holds for the point $p$.

From the condition (**) it follows that a family $\mathcal{P}$ consisting of open intervals $(p_1, p_2)$, where $p_1$ and $p_2$ are points defined as in (**), is a $\pi$-base on $R$. Since $R$ is complete, there is a point $x_0$ on $R$ such that the family $\mathcal{P}$ is the base at $x_0$. Now let $y > x_0$ be given. Since $Y$ is regular, there is a point $z$ such that $[x_0, z) = U(x_0, z) \subset U(x_0, y)$. From the fact that $\mathcal{P}$ is a base of $R$ at the point $x_0$ it follows that there are $p$ in $Y - S$ and $p_1, p_2$ in $S$ such that $(p_1, p_2) \subset c(x_0 - 1, z)$ and $x_0 \in (p_1, p_2)$ and the condition (**) holds.

From the condition (**) we infer that $p \in (x_0, z)$ and $p \in (x_0 - 1, x_0)$ (because $W \cap (x_0 - 1, x_0) \subset W \cap (p_1 - \varepsilon, p_2) \neq \emptyset$ and $W \cap [x_0, z) \subset W \cap (p_2 - \varepsilon, p_2) \neq \emptyset$ for each open neighbourhood $W$ of $p$ and for any $\varepsilon > 0$). But $p \in U(x_0, y)$ and $U(x_0, y)$ is the open neighbourhood of point $p$ such that $U(x_0, y) \cap [x_0 - 1, x_0) = [x_0, y) \cap [x_0 - 1, x_0) = \emptyset$. Hence $p \notin (x_0 - 1, x_0)$; a contradiction.

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3. Example. There exists a connected Hausdorff space $Y$ containing $S$ as a dense subset.

For each $x \in \mathbb{R}$, let $D_x = \{d_1, d_2, \ldots\}$ be an arbitrary sequence such that $d_i \in \mathbb{R}$, $d_i < d_{i+1} < x$ and $x = \lim_{i \to \infty} d_i$ for $i = 1, 2, \ldots$. By the Sierpiński's Theorem there exists a family $\mathcal{D}_x$ of the cardinality of continuum consisting of infinite subsets of $D_x$ the union which is $D_x$ and each two members of $\mathcal{D}_x$ have only finitely many points in common. Observe that each member of $\mathcal{D}_x$ is discrete and closed in $S$. Let $Z = A \times A$, where $A$ is an arbitrary subset of $S$ which is dense in $S$. By a transfinite induction we can define sets $D(x,y)$ of the form $K \cup L$, where $K \in \mathcal{D}_x$ and $L \in \mathcal{D}_y$, such that $D(x,y) \cap D(t,s)$ are finite or empty for $(x,y) \neq (t,s)$. Let $Y = S \cup Z$. Now we define the topology in $Y$. If $p \in S$, then let the collection of all subsets of $S$ of the form $\{p,x\}$ be a base in $Y$ at the point $p$. If $p = (x,y) \in Z$, then let the collection of all subsets $W$ of $Y$ of the form $W = \{p\} \cup G \{D_p - F\}$ be a base in $Y$ at the point $p$, where $F$ is a finite subset of $S$ and for each subset $B$ of $S$ $G[B]$ denotes an arbitrary open neighbourhood of the subset $B$ in $S$ and $D_p = D(x,y)$. Clearly, $S$ is a dense and open subspace of $Y$.

Now we prove that $Y$ is Hausdorff. If $p, q \in S$ and $p \neq q$, say $p < q$, then $\{p, q\}$ and $\{q, q + 1\}$ are two mutually disjoint open subsets in $Y$ containing $p$ and $q$. If $p, q \in Z$ and $p \neq q$, then $D_p \cap D_q = F$ is a finite subset of $S$. Hence $D_p - F$ and $D_q - F$ are closed and mutually disjoint subsets of $S$. Since $S$ is a normal space, there are mutually disjoint open subsets $G[D_p - F]$ and $G[D_q - F]$ in $S$. Hence $W_p = \{p\} \cup G[D_p - F]$
and $W_q = \{q\} \cup G [D_q - F]$ are mutually disjoint open subsets containing $p$ and $q$. If $p \in Z$ and $q \in S$, then also there are mutually disjoint open subsets $G [D_p - \{q\}]$ and $[q, q + \varepsilon)$ containing $p$ and $q$.

The space $Y$ is connected, because for each two mutually disjoint open subsets $U$ and $V$ of $Y$ there are points $x, y$ belonging to $A$ and $\varepsilon > 0$ such that $x \in (x - \varepsilon, x + \varepsilon) \subseteq U \cap S$, and $y \in (y - \varepsilon, y + \varepsilon) \subseteq V \cap S$ and therefore there is a point $p = (x, y)$ in $Y$ such that $p \in \overline{U \cap V}$.

**Remark.** If, in addition, the set $A$ defined above is a countable subspace of $S$ (for example the set $Q$ of rational numbers), then the space $Y$ is Lindelöf and a subspace $A \times A \cup A$ of $Y$ is an example of countable connected space.

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