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THE SORGENFREY LINE HAS NO CONNECTED COMPACTIFICATION

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Abstract: We answer the question raised by Eric van Douwen during the Conference at Štefanová, February 1977, whether there exists a connected compacfitication of the Sorgenfrey line. We prove that there is no regular Hausdorff connected space containing the Sorgenfrey line as a dense subspace. We give an example of Hausdorff connected space containing the Sorgenfrey line as a dense subspace.

Key words: Connected space, Sorgenfrey line.

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1. The statements of results. Let S be the Sorgenfrey line, i.e. the set R of reals with the topology generated by half-open intervals $[x,y)$ of R. If S is a subset of Y, then let $U[x,y)$ be the greatest open subset of Y such that $U[x,y) \cap S = [x,y)$.

Theorem. There exists no regular space Y such that S is a dense subspace of Y and

(*) $\overline{[x,y)} \cap \overline{S - [x,y)} \neq \emptyset$ for each $[x,y) \subset S$.

Remark. If Y is a connected space, or if Y - S is connected and Y is compact, then the condition (*) holds.

Corollary 1. There exists no regular connected Hausdorff space containing S as a dense subset.

Corollary 2. There exists no compactification of S with a connected remainder.

2. The proofs. We begin from a

Lemma. Let Y be a Hausdorff space containing S as a dense subset. If $p \in \overline{[x,y)} \cap (Y - S)$, then there exists a q in S such that $x < q \leq y$ and such that for each open neighbourhood W of p the open interval $(q - \varepsilon, q)$ intersects W for each $\varepsilon > 0$.

Proof of Lemma: Let $q = \sup \{ r \in S$: there exists an open neighbourhood W of p such that $W \cap [x,r) = \emptyset \}$. Since Y is Hausdorff, there is an open neighbourhood W of p and there is a point r in S such that $[x,r) \cap W = \emptyset$. Therefore $q > x$. If $q > y$, then there are $r > y$ and an open neighbourhood W of p such that $[x,r) \cap W = \emptyset$. This contradicts the fact that $p \in \overline{[x,y)}$. Hence $q \leq y$. It remains to show that $W \cap (q - \varepsilon, q) \neq \emptyset$ for each $\varepsilon > 0$ and for each open neighbourhood W of p. Suppose not. Then there are $\varepsilon > 0$ and an open neighbourhood W_1 of p such that $W_1 \cap (q - \varepsilon, q) = \emptyset$. From the definition of point q there is an open neighbourhood W_2 of p such that $W_2 \cap [x, q - \frac{\varepsilon}{2}) = \emptyset$. Since Y is Hausdorff, there is an open neighbourhood W_3 of p such that $W_3 \cap [q, q + \varepsilon_1) = \emptyset$ for any $\varepsilon_1 > 0$. Hence $W \cap [x, q + \varepsilon_1) = \emptyset$ where $W = W_1 \cap W_2 \cap W_3$. This contradicts the definition of point q.

Proof of the Theorem. Suppose that there is a regular Hausdorff space Y containing S as a dense subset and the condition (*) holds. We first show that

(**) for each x, y in S there exist p in Y - S and p_1, p_2 in $(x,y]$ such that $p_1 \neq p_2$ and $W \cap (p_1 - \varepsilon, p_1) \neq \emptyset$ and

$W \cap (p_2 - \varepsilon, p_2) \neq \emptyset$ for each $\varepsilon > 0$ and open neighbourhood W of p.

Since Y is regular, there is a point z in (x,y) such that $\overline{[x,z)} \subset U[x,y)$. From the condition $(*)$ there is a point p in $\overline{[x,z)} \cap \overline{S - [x,z)}$. Then $p \in \overline{[x,z)} \subset U[x,y)$ and $p \in \overline{S - [x,z)}$, and therefore for each open neighbourhood W of p we have $\emptyset \neq W \cap U[x,y) \cap (S - [x,z)) = W \cap [x,y) \cap (S - [x,z)) = W \cap$ $\cap [z,y)$. This implies that $p \in \overline{[z,y)}$. Hence there exists a point p in Y - S belonging to $\overline{[x,z)}$ and $\overline{[z,y)}$. By the Lemma, there exist p_1 and p_2 in S such that $(*)$ holds for the point p.

From the condition $(**)$ it follows that a family $\mathcal{P}$ consisting of open intervals (p_1, p_2), where p_1 and p_2 are points defined as in $(**)$, is a π-base on R. Since R is complete, there is a point x_0 on R such that the family $\mathcal{P}$ is the base at x_0. Now let $y > x_0$ be given. Since Y is regular, there is a point z such that $\overline{[x_0,z)} = \overline{U[x_0,z)} \subset U[x_0,y)$. From the fact that $\mathcal{P}$ is a base of R at the point x_0 it follows that there are p in Y - S and p_1, p_2 in S such that $(p_1,p_2) \subset$ $\subset (x_0 - 1, z)$ and $x_0 \in (p_1,p_2)$ and the condition $(**)$ holds. From the condition $(**)$ we infer that $p \in \overline{[x_0,z)}$ and $p \in$ $\in \overline{[x_0 - 1, x_0)}$ (because $W \cap [x_0 - 1, x_0) \supset W \cap (p_1 - \varepsilon, p_1) \neq \emptyset$ and $W \cap [x_0,z) \supset W \cap (p_2 - \varepsilon, p_2) \neq \emptyset$ for each open neighbourhood W of p and for any $\varepsilon > 0$). But $p \in U[x_0,y)$ and $U[x_0,y)$ is the open neighbourhood of point p such that $U[x_0,y) \cap [x_0 -$ $- 1, x_0) = [x_0,y) \cap [x_0 - 1, x_0) = \emptyset$. Hence $p \notin \overline{[x_0 - 1, x_0)}$; a contradiction.

3. <u>Example</u>. There exists a connected Hausdorff space Y containing S as a dense subset.

For each $x \in R$, let $D_x = \{d_1, d_2, \dots\}$ be an arbitrary sequence such that $d_i \in R$, $d_i < d_{i+1} < x$ and $x = \lim_{i \to \infty} d_i$ for $i = 1, 2, \dots$. By the Sierpiński's Theorem there exists a family $\mathcal{D}_x$ of the cardinality of continuum consisting of infinite subsets of D_x the union which is D_x and each two members of $\mathcal{D}_x$ have only finitely many points in common. Observe that each member of $\mathcal{D}_x$ is discrete and closed in S. Let $Z = A \times A$, where A is an arbitrary subset of S which is dense in S. By a transfinite induction we can define sets $D(x,y)$ of the form $K \cup L$, where $K \in \mathcal{D}_x$ and $L \in \mathcal{D}_y$, such that $D(x,y) \cap D(t,s)$ are finite or empty for $(x,y) \neq (t,s)$. Let $Y = S \cup Z$. Now we define the topology in Y. If $p \in S$, then let the collection of all subsets of S of the form $[p,x)$ be a base in Y at the point p. If $p = (x,y) \in Z$, then let the collection of all subsets W of Y of the form $W = \{p\} \cup G[D_p - F]$ be a base in Y at the point p, where F is a finite subset of S and for each subset B of S $G[B]$ denotes an arbitrary open neighbourhood of the subset B in S and $D_p = D(x,y)$. Clearly, S is a dense and open subspace of Y.

Now we prove that Y is Hausdorff. If $p, q \in S$ and $p \neq q$, say $p < q$, then $[p,q)$ and $[q, q+1)$ are two mutually disjoint open subsets in Y containing p and q. If $p, q \in Z$ and $p \neq q$, then $D_p \cap D_q = F$ is a finite subset of S. Hence $D_p - F$ and $D_q - F$ are closed and mutually disjoint subsets of S. Since S is a normal space, there are mutually disjoint open subsets $G[D_p - F]$ and $G[D_q - F]$ in S. Hence $W_p = \{p\} \cup G[D_p - F]$

and $W_q = \{q\} \cup G[D_q - F]$ are mutually disjoint open subsets containing p and q. If $p \in Z$ and $q \in S$, then also there are mutually disjoint open subsets $G[D_p - \{q\}]$ and $[q, q+\varepsilon)$ containing p and q.

The space Y is connected, because for each two mutually disjoint open subsets U and V of Y there are points x, y belonging to A and $\varepsilon > 0$ such that $x \in (x - \varepsilon, x + \varepsilon) \subset U \cap S$ and $y \in (y - \varepsilon, y + \varepsilon) \subset V \cap S$ and therefore there is a point $p = (x,y)$ in Y such that $p \in \overline{U} \cap \overline{V}$.

<u>Remark</u>. If, in addition, the set A defined above is a countable subspace of S (for example the set Q of rational numbers), then the space Y is Lindelöf and a subspace $A \times A \cup A$ of Y is an example of countable connected space.

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