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SEQUENTIAL REGULARIZATION OF ILL-POSED PROBLEMS INVOLVING UNBOUNDED OPERATORS

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Abstract: Let \( A: D(A) \rightarrow H \) be a closed densely defined linear operator in a real Hilbert space \( H \) and suppose that for a certain \( f \in H \) the ill-posed problem \( Au = f \) has a unique solution \( u \). Let \( B \) be a bounded positive definite operator on \( H \) and set \( u_0 = 0 \). Then for \( n = 1, 2, \ldots \) the well-posed problem

\[
\langle Au_n, Av \rangle + \langle Bu_n, v \rangle = \langle Bu_{n-1}, v \rangle + \langle f, Av \rangle, \quad \forall v \in D(A)
\]

has a unique solution \( u_n \in D(A) \) and \( u_n \rightarrow u \) as \( n \rightarrow \infty \).

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1. Introduction. Suppose that \( H \) is a real Hilbert space and \( D(A) \subset H \) is a dense subspace. This paper is a theoretical study of a method of approximating the solution of the problem

\[
(1) \quad Au = f
\]

where \( f \in H \) and \( A: D(A) \rightarrow H \) is a closed unbounded operator. We assume that for a certain \( f \in H \) the problem (1) has a unique solution \( u \), without assuming that \( A \) is an isomorphism of \( D(A) \) onto the range of \( A \). It is then well-known that equation (1) is ill-posed, that is, for small perturbations of the equation
Ax = f + \delta f

may have no solution at all, or may have a solution x which is not near to the solution u of equation (1). We will show that the solution of (1) may be approximated by a sequence of solutions of associated well-posed problems. The idea of replacing a problem of type (1) by a family of nearby well-posed problems has been studied extensively by Lattes and Lions [3] under the title "quasi-reversibility". In particular Lattes and Lions [3, p. 289] show that the problem

(2) \langle Au_\varepsilon, Av \rangle + \varepsilon \langle u_\varepsilon, v \rangle = \langle f, Av \rangle, \quad \forall v \in D(A)

is well-posed for each \varepsilon > 0 and the solutions u_\varepsilon of (2) converge to the solution u of (1) as \varepsilon \to 0. In solving (2) one must in essence "invert the operator \varepsilon I + A^* A ", which depends on the parameter \varepsilon . In this paper we will replace equation (1) by a sequence of well-posed problems the solution of which requires the inversion of a single operator which is independent of the parameter. The method considered here is related to an analogous procedure for bounded operators studied by Kryanev [2].

As an example of a specific problem of type (1) Lattes and Lions [3, p. 290] consider the boundary value problem

\[ Au = 0 \]
\[ u \big|_{\Gamma_0} = g_0 \]
\[ \frac{\partial u}{\partial \nu} \bigg|_{\Gamma_0} = g_1 \quad \text{(conormal derivative)} \]

where \Gamma_0 is the boundary of an open domain \Omega \subset \mathbb{R}^2 and A
is a second order differential operator in $\Omega$ given by

$$Au = -\sum_{i,j=1}^{n} \sum_{\alpha=1}^{\infty} a_{ij}(x) \frac{\partial u}{\partial x_{i}} \frac{\partial u}{\partial x_{j}} + a_{0}u$$

where $a_{ij} \in C^{3}(\overline{\Omega})$, $a_{0} \in C^{0}(\overline{\Omega})$, $a_{ij}(x) \xi_{i} \xi_{j} \geq \alpha_{1}(\xi_{1}^{2} + \ldots + \xi_{n}^{2})$, $\alpha_{1} > 0$

and

$$\alpha_{0}(x) \geq \alpha_{0} > 0.$$

This problem is analyzed by finding a function $\Phi \in H^{2}(\Omega)$ such that

$$\Phi|_{\Gamma_{0}} = \xi_{0}, \quad \frac{\partial \Phi}{\partial \nu_{A}}|_{\Gamma_{0}} = \xi_{1}$$

and considering the problem satisfied by $w = u - \Phi$:

$$Aw = f$$

$$w|_{\Gamma_{0}} = 0$$

$$\frac{\partial w}{\partial \nu_{A}}|_{\Gamma_{0}} = 0$$

where $f = -A\Phi$. The domain of the unbounded operator $A$ is then given by

$$D(A) = \{v \in L^{2}(\Omega) : Av \in L^{2}(\Omega), \quad v|_{\Gamma_{0}} = 0, \quad \frac{\partial v}{\partial \nu_{A}}|_{\Gamma_{0}} = 0\}.$$

For details the reader is referred to Lattes and Lions [3].

\[ Bx_n + Ax_n = Bx_{n-1} + f \]

for approximating solutions to the ill-posed problem
\[ Ax = f \]

where \( A \) is a bounded positive semi-definite linear operator on a Hilbert space \( H \) and \( B \) is a bounded positive definite operator on \( H \) which is chosen to improve the conditioning of the operator \( B + A \). However, as noted above, many ill-posed problems which are of practical interest may be formulated as an equation of type (1) where \( A \) is a closed, densely defined but unbounded operator on a suitable Hilbert space.

We will examine Kryanev's procedure in the context considered by Lattes and Lions. Below, \( A:D(A) \rightarrow H \) will be a closed linear operator defined on the dense subspace \( D(A) \) of the real Hilbert space \( H \) and \( B \) will be a bounded linear operator on \( H \) satisfying

\[ \langle Bx, x \rangle \geq c \| x \|^2, \quad c > 0. \]

We recall that the domain \( D(A^*) \) of the adjoint operator is by definition the set of all vectors \( y \in H \) for which there is a \( y^* \in H \) satisfying

\[ \langle Ax, y \rangle = \langle x, y^* \rangle, \quad \forall x \in D(A) \]

and the adjoint operator \( A^* \) is defined by \( A^* y = y^* \).

First we state a lemma which will be useful in the sequel.

Lemma 1. The operator \( B + A^* A \) has a bounded inverse \( U = (B + A^* A)^{-1}:H \rightarrow D(A^* A) \) which is positive.

Proof. By assumption there is a number \( c > 0 \) such that
\[ \langle Bx, x \rangle \geq c \| x \| ^2 \] for each \( x \in H \). Choose \( k > 0 \) such that \[ \max \{ |kc - 1|, k \| B \| \} < 1. \] Let \( \bar{A} = kA \), then by a theorem in Riesz and Sz.-Nagy [5, p. 307], \( (I + \bar{A} \bar{A})^{-1} : H \to D(A^*A) \) exists and \( \| (I + \bar{A} \bar{A})^{-1} \| \leq 1 \). Now,

\[ \| (kB - \bar{A} \bar{A}) - (I + \bar{A} \bar{A}) \| \leq \max \{ |kc - 1|, k \| B \| \} < 1, \]

and it follows by a standard perturbation result (see e.g. [4, p. 45]) that

\[ kB + \bar{A} \bar{A} = k(B + A^*A) \]

is invertible. Hence \( B + A^*A \) is invertible and it can be shown that \( \langle (B + A^*A)^{-1}x, x \rangle \geq 0 \) for all \( x \in H \) as in [5, p. 308].

The next lemma defines a sequence of well-posed problems the solutions of which we shall show converge to the solution \( u \) of equation (1).

**Lemma 2.** Set \( u_0 = 0 \), then for \( n = 1, 2, \ldots \), the problem

\[ (3) \quad \langle Bu_n, v \rangle + \langle Au_n, Av \rangle = \langle Bu_{n-1}, v \rangle + \langle f, Av \rangle, \quad \forall v \in D(A) \]

has a unique solution \( u_n \in D(A) \) which depends continuously on \( f \).

**Proof.** Since \( A \) is a closed linear operator, the subspace \( D(A) \) endowed with the norm

\[ \| x \| _{D(A)} = (\| x \|^2 + \| Ax \|^2)^{1/2} \]

and corresponding inner product

\[ \langle x, y \rangle _{D(A)} = \langle x, y \rangle + \langle Ax, Ay \rangle \]

is a Hilbert space. Define the symmetric bilinear form \( Q(x, y) \) on \( D(A) \times D(A) \) by

\[ Q(x, y) = \langle Bx, y \rangle + \langle Ax, Ay \rangle. \]

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It is easy to see that $Q(x,y)$ is continuous (with respect to the norm $\| \cdot \|_{D(A)}$) and for $x \in D(A)$

$$Q(x,x) \geq \min(c,1) \|x\|^2_{D(A)}.$$ 

Hence $Q(x,y)$ is coercive and the existence of $u_n$ follows by use of the Lax-Milgram lemma (see e.g. [1, p.41]). Furthermore, if

$$Q(u_n, v) = \langle Bu_{n-1}, v \rangle + \langle f, Av \rangle, \quad \forall v \in D(A)$$

and

$$Q(u_n', v) = \langle Bu_{n-1}, v \rangle + \langle f', Av \rangle, \quad \forall v \in D(A),$$

then setting $v = u_n - u_n'$, we obtain

$$\min(c,1) \| u_n - u_n' \|_{D(A)}^2 \leq Q(u_n - u_n', u_n - u_n')$$

$$= \langle f - f', A(u_n - u_n') \rangle$$

$$\leq \| f - f' \|_{D(A)} \| u_n - u_n' \|_{D(A)}.$$

From this it follows that $u_n$ is unique and the mapping $f \mapsto u_n$ is continuous.

The main result may now be stated.

**Theorem.** The solutions $u_n$ of the well-posed problems (3) converge strongly to the solution $u$ of problem (1).

Before proceeding with the proof we note that

(4) $$u_n = UBu_{n-1} + u_1$$

where $U = (B + A^* A)^{-1}$. In fact, we have by Lemma 2

$$\langle A(UBu_{n-1}) + u_1, Av \rangle + \langle B(UBu_{n-1} + u_1), v \rangle$$

$$= \langle AUBu_{n-1}, Av \rangle + \langle BUBu_{n-1}, v \rangle + \langle f, Av \rangle =$$

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Equation (4) now follows by the uniqueness statement in Lemma 2. The proof of the theorem requires two further lemmas.

Lemma 3. For $m = 1,2,\ldots$, $\langle Bu_m, u_m - u \rangle \leq 0$.

Proof. Note that by Lemma 2 and equation (1), we have for all $v \in D(A)$

$$\langle Bu_m, v \rangle + \langle A \sum_{n=1}^{m} (u_n - u), Av \rangle = \sum_{n=1}^{m} \langle B(u_n - u_{n-1}), v \rangle + \langle A(u_n - u), Av \rangle \geq 0,$$

Hence it suffices to show that

$$(5) \quad \langle A \sum_{n=1}^{m} (u_n - u), A(u_m - u) \rangle \geq 0, \quad m = 1,2,\ldots .$$

Note that

$$\langle A(u - UBu), Av \rangle + \langle B(U - UBu), v \rangle = \langle f, Av \rangle - \langle (A^* A + B)UBu - Bu, v \rangle = \langle f, Av \rangle, \quad \forall v \in D(A)$$

and it follows from Lemma 2 that

$$(6) \quad u_1 = u - Wu$$

where $W = UB$. We therefore have by (4) and (6)

$$W(u_{m-1} - u) = Wu_{m-1} + u_1 - u = u_m - u,$$

and hence for $j \leq m$ we have
Therefore, for $j < m$

$$\langle A(u_m - u), A(u_j - u) \rangle = \langle A W^{m-j} (u_j - u), A(u_j - u) \rangle$$

$$= \langle A * A W^{m-j} (u_j - u), u_j - u \rangle.$$

But

$$\langle A * A W^k x, x \rangle = \langle A * A W^k x, (I + B^{-1} A * A)^k W^k x \rangle$$

$$= \langle A * A W^k x, W^k x \rangle + \sum_{j=1}^{k} \langle A * A W^k x, (B^{-1} A * A)^{j-1} B^{-1} A * A W^k x \rangle,$$

and it is easy to show that $(B^{-1} A * A)^n B^{-1}$ is positive for $n = 0, 1, 2, \ldots$, and hence $\langle A(u_m - u), A(u_j - u) \rangle \geq 0$, which proves the lemma.

From the above lemma it follows that the sequence $\{u_n\}$ is bounded, indeed

$$(7) \quad e \|u_n\|^2 \leq \langle B u_n, u_n \rangle \leq \langle B u_n, u \rangle \leq \|B\| \|u_n\| \|u\|.$$

**Lemma 4.** As $n \to \infty$, $A u_n \to A u$.

Proof. Setting $v = u_n - u$ in the equation

$$\langle A(u_n - u), A v \rangle = \langle B(u_{n-1} - u_n), v \rangle$$

and summing we obtain

$$(8) \quad \sum_{n=1}^{m} \|A(u_n - u)\|^2 = \langle B u_m, u \rangle - \sum_{n=1}^{m} \langle B(u_n - u_{n-1}), u_n \rangle.$$

If we define a new inner product and norm by

$$(x, y) = \langle B x, y \rangle \quad \text{and} \quad \|x\|_B^2 = (x, x),$$

then

$$\sum_{n=1}^{m} \langle B(u_n - u_{n-1}), u_n \rangle = \sum_{n=1}^{m} \{ \|u_n\|_B^2 - (u_{n-1}, u_n)^2 \}$$

$$= \frac{\|u_1\|_B^2}{2} + \frac{\|u_m\|_B^2}{2} +$$

$$- 496 -$$
Therefore
\[ \frac{1}{2} \sum_{n=2}^{\infty} \left\{ \| u_n \| B^2 - 2(u_{n-1}, u_n) + \| u_{n-1} \| B^2 \right\}. \]

Therefore
\[ \sum_{n=1}^{\infty} \langle B(u_n - u_{n-1}), u_n \rangle \geq 0 \]

and it follows from (8) and (7) that
\[ \sum_{n=1}^{\infty} \| A(u_n - u) \| B^2 \leq \langle B u_n, u \rangle \leq \| B \| \| u \| B^2/c, \]

which proves the lemma.

Finally we are in a position to complete the proof of the theorem. Since any subsequence of \( \{ u_n \} \) is bounded, we can extract a subsequence which converges weakly to an element \( z \in H \). Since the graph of \( A \) is closed and convex, it is weakly closed and therefore \( z \in D(A) \) and from Lemma 4 we have \( Az = Au = f \). But the solution to problem (1) is unique, therefore \( z = u \). Hence we see that any subsequence of \( \{ u_n \} \) in turn contains a subsequence which converges weakly to \( u \). It follows that the entire sequence \( \{ u_n \} \) converges weakly to \( u \). By Lemma 3 we have \( \langle B u_n, u_n - f \rangle \leq 0 \) and hence
\[
eq \langle B u_m, u_m - u \rangle \leq \langle B u_m, u_m - u \rangle \leq \langle B u_m, u_m - u \rangle \rightarrow 0.
\]

Therefore, \( u_n \rightarrow u \), completing the proof of the theorem.

References


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