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RING VARIETIES CLOSED UNDER IDEAL SUMS

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Abstract: A variety $\mathcal{V}$ of rings or algebras is S-closed if $I + J \in \mathcal{V}$ whenever I and J are ideals of a ring or algebra A and both I and $J \in \mathcal{V}$. A variety of associative algebras over a field is S-closed if and only if it is closed under extensions. A non-trivial S-closed variety $\mathcal{V}$ of associative rings can contain no rings with torsion-free additive groups and consequently $\mathcal{V}$ is determined by subvarieties $\overline{\mathcal{V}}_p = \{A \in \mathcal{V} \mid$ The additive group of A is p-primary$\}$ where p is prime. For almost all p, $\overline{\mathcal{V}}_p = \{0\}$; otherwise, either $\overline{\mathcal{V}}_p$ is extension-closed or $\overline{\mathcal{V}}_p$ is between the varieties defined by $p^{[m/2]}x = 0$, $p^m x = 0$ for some positive integer m.

Key words: Variety, associative ring.

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Introduction. We shall call a variety $\mathcal{V}$ of rings or algebras S-closed if whenever I, J are ideals of a ring or algebra A, with I, $J \in \mathcal{V}$, we also have $I + J \in \mathcal{V}$. Examples of S-closed varieties are the extension-closed varieties, those varieties $\mathcal{V}$ with the property that if a ring or algebra A has an ideal J with J and $A/J \in \mathcal{V}$, then $A \in \mathcal{V}$. (Of course, in both of these definitions everything is assumed to be happening in some prescribed "universal" variety $\mathcal{U}$ and, more precisely, the varieties in which we are interested are the subvarieties of $\mathcal{U}$ which have the properties relative to $\mathcal{U}$.) For if $\mathcal{V}$ is extension-closed and I, J are two ideals of a ring or algebra A, then $(I + J)/J \cong I/I \cap J$, so $I + J \in \mathcal{V}$ if $I, J \in \mathcal{V}$.

Note that an S-closed variety $\mathcal{V}$ is closed under formation of arbitrary ideal sums. For if $\{I_\lambda \mid \lambda \in \Lambda\}$ is a set of ideals from $\mathcal{V}$ and $a_1,\dots,a_n \in \sum I_\lambda$, then $\{a_1,\dots$ $\dots,a_n\} \subseteq I_{\lambda_1} + \dots + I_{\lambda_n} \in \mathcal{V}$ for some $\lambda_1,\dots,\lambda_n \in \Lambda$. Thus the subring generated by $\{a_1,\dots,a_n\}$ is in $\mathcal{V}$. This being so for any finite set, we have $\sum I_\lambda \in \mathcal{V}$.

Essentially we shall be concerned with associative rings and algebras. In § 1 we show that for algebras over a field, a variety is S-closed if and only if it is extension-closed. The situation is not as simple for rings. For instance, for any positive integer ℓ, the variety defined by the identity $\ell x = 0$ is clearly S-closed. Other examples can be obtained by combining varieties like the one just mentioned with extension-closed varieties. Whether or not all S-closed varieties arise in this way is still unknown, but we do show that for any non-trivial S-closed variety $\mathcal{V}$, there are finitely many primes $p_1,\dots,p_k$ such that every ring in $\mathcal{V}$ is a direct sum of rings whose characteristics are powers of $p_1,\dots,p_k$, and for each i, either there are integers $1 \leq$ $\leq [m/2] \leq n \leq m$ such that $\overline{\mathcal{V}}_p = \{A \in \mathcal{V} \mid A$ has a p_i-primary additive group$\}$ is between the varieties defined by $p_i^n x = 0$ and $p_i^m x = 0$, or $\overline{\mathcal{V}}_p$ is an extension-closed variety.

We shall use the following notation: A^+ is the additive group of a ring or algebra A, A^{+0} the zeroring (or zero-algebra) on A^+ (i.e. the ring or algebra with all products zero); for elements $u_1,\dots,u_n$ of a ring, $\langle u_1,\dots,u_n\rangle$ is the subring they generate.

The group theory analogue of the problem treated here was solved by T.S. Shores [7] : there are no non-trivial varieties of groups closed under normal products. The author is grateful to Professor Shores for calling his attention to this result and to the associated question for ring varieties.

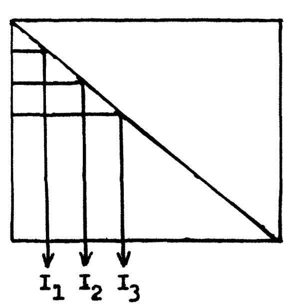
1. <u>The Algebra Case</u>. Let Ω be a commutative ring with identity. We first consider S-closed varieties of Ω-algebras containing Ω^{+0}.

<u>Proposition 1.1.</u> <u>If</u> Ω^{+0} <u>belongs to an</u> S-<u>closed variety</u> $\mathcal{V}$ <u>of</u> Ω <u>-algebras, then</u> $\mathcal{V}$ <u>contains all</u> Ω <u>-algebras</u>.

<u>Proof</u>. Consider the algebra $T_n(\Omega)$ of strictly lower triangular $n \times n$ matrices over Ω for a fixed integer $n > 1$. For $k = 1, 2, \ldots, n-1$, let

$$I_k = \{(a_{ij}) \in T_n(\Omega) \mid a_{ij} = 0 \text{ if } i < k+1 \text{ or } j > k\} .$$

The I_k are depicted in the following diagram



Each I_k is an ideal of $T_n(\Omega)$ and $I_k^2 = 0$ for each k, so, by assumption, $\mathcal{V}$ contains $I_1 + I_2 + \ldots + I_{n-1} = T_n(\Omega)$. This is so for every value of n, so $\prod_{m>1} T_n(\Omega) \in \mathcal{V}$. Then if $\mathcal{V}$ is not the class of all algebras, $\prod T_n(\Omega)$ satisfies a polynomial identity and hence (cf. [6], pp. 181-182) a homogeneous multilinear identity $\sum_{\sigma} a_{\sigma} x_{\sigma(1)} \cdots x_{\sigma(m)} = 0$ where σ varies over some set of permutations of $\{1,2,\ldots,m\}$ and each $a \in \Omega \setminus \{0\}$. Now each $T_n(\Omega)$ satisfies this identity; in particular this is so when $n > m$. For such an n , consider the matrices $E_{21}, E_{32}, \ldots, E_{m,m-1} \in T_n(\Omega)$, where E_{ij} has 1 in the (i,j) position and zeros elsewhere. We have $E_{m,m-1} E_{m-1,m-2} \cdots E_{32} E_{21} = E_{m1}$, while the product taken in any other order is zero. But then $a_{\sigma} E_{n1} = 0$ for some σ , which clearly is impossible. Thus there is no proper identity satisfied by $\mathcal{V}$, i.e. $\mathcal{V}$ contains all algebras.//

The case where Ω is a field is worthy of separate mention.

Proposition 1.2. *If Ω is a field and $\mathcal{V}$ is an S-closed variety of Ω -algebras, then $\mathcal{V}$ is either*

(i) $\{0\}$,

(ii) *the class of all algebras, or*

(iii) *an extension-closed variety.*

Hence if Ω is infinite, $\mathcal{V}$ must be (i) *or* (ii) *and if Ω is finite, $\mathcal{V}$ is* (i),(ii) *or the variety generated by a finite set of finite extension fields of Ω* .

Proof. If $\mathcal{V}$ is neither (i) nor (ii), then by Proposition 1.1, $\Omega^{+0} \notin \mathcal{V}$ and $\mathcal{V}$ therefore contains no algebra with a non-zero nilpotent element. That $\mathcal{V}$ is extension-closed can now be proved by analogy with the theorem of [3] and other results quoted in that paper. The final assertion follows from results in § 2 of [1] . //

2. The Ring Case. Throughout this section, $\mathcal{V}$ will always be an S-closed variety of rings. We proceed to our principal result by a sequence of propositions. Let $\mathcal{V}_p = \{A \in \mathcal{V} \mid pA = 0\}$ for each prime p.

Proposition 2.1. If $\mathcal{V}_p \neq \{0\}$ for infinitely many primes p, then $\mathcal{V}$ contains all rings.

Proof. Note firstly that $\mathcal{V}_p$ can be viewed as an S-closed variety of algebras over the field K_p of p elements, so that by Proposition 1.1, $\mathcal{V}_p$ contains all rings of characteristic p if $K_p^{+0} \in \mathcal{V}_p$. Let $P = \{p \mid K_p^{+0} \in \mathcal{V}_p\}$, $M = \{p \mid \mathcal{V}_p \neq \{0\},\ K_p^{+0} \notin \mathcal{V}_p\}$.

If P is infinite, consider a free ring F on $\aleph_0$ generators. By Proposition 1.1, $F/pF \in \mathcal{V}_p$ for each $p \in P$. But $\bigcap_p pF = 0$, so F, as a subdirect product of $\{F/pF \mid p \in P\}$, is in $\mathcal{V}$, and therefore $\mathcal{V}$ contains all rings. If, on the other hand, M is infinite, then by Corollary 1.2, there is a field $L_p \in \mathcal{V}_p$ for each $p \in M$ and then $\prod_M L_p \in \mathcal{V}$. The element e of $\prod L_p$ whose p-component is the identity of L_p for each p is idempotent and has infinite additive order. It follows that $\mathcal{V}$ contains the ring Z of integers. But then $K_q^{+0} = qZ/q^2Z \in \mathcal{V}$ for *every* prime q and so, as before, $\mathcal{V}$ contains all rings. //

Proposition 2.2. If $\mathcal{V}$ contains a ring A for which A^+ is torsion-free, then $\mathcal{V}$ contains all rings.

Proof. Let $A \in \mathcal{V}$ have torsion-free additive group. First suppose there is an $a \in A$ with $a^2 = 0 \neq a$. Then $K_p^{+0} \cong \langle a \rangle / p\langle a \rangle \in \mathcal{V}$ for every prime p, so by Proposition 2.1, $\mathcal{V}$ contains all rings.

Now suppose A has no non-zero nilpotent elements. If $0 \neq b \in A$, then $\langle b \rangle$ has no nil ideals, so neither does $\langle b \rangle^*$, the ring obtained from $\langle b \rangle$ by the adjunction of an identity element in the usual way. But $\langle b \rangle^*$ is a homomorphic image of $Z[x]$, so $\langle b \rangle^*$ has nil Jacobson radical ([4], Theorems 2 and 3), whence $\langle b \rangle^*$, and therefore also $\langle b \rangle$, is semiprimitive. We can thus represent $\langle b \rangle$ as a subdirect product of a family $\{D_i \mid i \in I\}$ of primitive rings. Since b has infinite additive order, either

(i) some D_i has characteristic 0, or

(ii) the D_i have infinitely many different characteristics.

In case (i), let D_j have characteristic 0. Then there is a division ring Δ such that either $D_j \cong M_m(\Delta)$, the ring of $m \times m$ matrices over Δ, for some m, or for every n, D_j has a subring B_n with $M_n(\Delta)$ as a homomorphic image. (See [5], pp. 43-44.) In any case, $\mathcal{V}$ contains D_j, hence some $M_m(\Delta)$ and therefore Δ. But Δ has a subring $\cong Z$ so as in the proof of Proposition 2.1, we see that $\mathcal{V}$ contains all rings. In case (ii), arguing as for case (i) but using D_i's of various finite characteristics, we can show that $K_p \in \mathcal{V}$ for infinitely many primes p. By Proposition 2.1, $\mathcal{V}$ again contains all rings.

<u>Corollary 2.3</u>. <u>If $\mathcal{V}$ is not the class of all rings, then $\mathcal{V}$ consists of torsion rings and the set of orders of elements of rings in $\mathcal{V}$ is finite</u>.

<u>Proof</u>. If $\ell_1, \ell_2, \ldots$ are infinitely many distinct orders, choose, for each k, a ring $A_k \in \mathcal{V}$ containing an element of order ℓ_k. Then $\mathcal{V}$ contains $\prod_k A_k$, and hence

also its (non-zero) torsion-free factor ring, contradicting Proposition 2.2. //

Thus there is a finite set $\{p_1,\dots,p_k\}$ of primes, and a finite set $\{r_1,\dots,r_k\}$ of positive integers such that every ring R has a unique representation

$$R = R_1 \oplus \dots \oplus R_k,$$

where $p_i^{\,r_i}R_i = 0$ for each i. This of course is true of any variety consisting of torsion rings. For such a variety $\mathcal{U}$, let

$$\overline{\mathcal{U}}_p = \{A \in \mathcal{U} \mid A^+ \text{ is a p-group}\},$$

for all primes p. Among the varieties consisting of torsion rings, there are the classes $\{A \mid \ell A = 0\}$ for all positive integers ℓ. These are clearly S-closed. As mentioned in the introduction, so are the non-trivial extension-closed varieties - the varieties generated by finite sets of finite fields.

We can now state our principal result.

<u>Theorem 2.4</u>. <u>Let $\mathcal{V}$ be a non-trivial S-closed variety of rings. Then $\mathcal{V}$ consists of torsion rings, and those $\overline{\mathcal{V}}_p \neq \{0\}$ are described by the following conditions.</u>

(i) <u>If $\overline{\mathcal{V}}_p$ contains no nilpotent rings, then $\overline{\mathcal{V}}_p$ is the variety generated by a finite set of finite fields.</u>

(ii) <u>If $\overline{\mathcal{V}}_p$ contains a nilpotent ring, there exist positive integers</u> m(p), n(p) <u>such that</u>

(1) $\{A \mid p^{n(p)} A = 0\} \subseteq \overline{\mathcal{V}}_p \subseteq \{A \mid p^{m(p)} A = 0\}$;

(2) $p^{n(p)} A = 0$ <u>for every</u> $A \in \overline{\mathcal{V}}_p$ <u>with</u> $A^2 = 0$;

(3) $[m(p)/2] \leq n \leq m$.

On the other hand, a variety $\mathcal{U}$ consisting of torsion rings is S-closed if those $\overline{\mathcal{U}}_p \neq \{0\}$ are described by (i) and

(iii) If $\overline{\mathcal{U}}_p$ contains a nilpotent ring, then there exists a positive integer n(p) such that $\overline{\mathcal{U}}_p = \{A \mid p^{n(p)} A = 0\}$.

Proof. If $\mathcal{V}$ is S-closed, then by Corollary 2.3, $\mathcal{V}$ consists of torsion rings. Clearly each $\overline{\mathcal{V}}_p$ is an S-closed variety. If $\overline{\mathcal{V}}_p \neq \{0\}$ and there are no nilpotent rings in $\overline{\mathcal{V}}_p$, then $\overline{\mathcal{V}}_p$ is generated by a finite set of finite fields (cf. [3]). If there are nilpotent rings in $\overline{\mathcal{V}}_p$, let

$$n(p) = \mathrm{Max}\{k \mid \mathcal{V} \text{ contains a ring } A \text{ with } A^2 = 0 \text{ and } p^k A = 0 \neq p^{k-1} A\},$$

$$m(p) = \mathrm{Max}\{k \mid \mathcal{V} \text{ contains a ring } A \text{ with } p^k A = 0 \neq p^{k-1} A\}.$$

Let Ω be the ring of integers modulo $p^{n(p)}$. Then

$$\overline{\mathcal{V}}_p \cap \{A \mid p^{n(p)} A = 0\}$$

can be viewed as an S-closed variety of Ω-algebras, containing Ω^{+0} and therefore, by Proposition 1.1, all Ω-algebras. Hence

$\{A \mid p^{n(p)} A = 0\} \subseteq \overline{\mathcal{V}}_p$. Also, $\overline{\mathcal{V}}_p \subseteq \{A \mid p^{m(p)} A = 0\}$.

Let R be a ring in $\mathcal{V}$ with $p^{m(p)}R = 0 \neq p^{m(p)-1}R$. Let

$$t = [(m(p)-1)/2] + 1 = \begin{cases} (m(p)+1)/2 & \text{if } m(p) \text{ is odd} \\ m(p)/2 & \text{if } m(p) \text{ is even} \end{cases}.$$

Then $(p^t R)^2 \subseteq p^{m(p)}R = 0$. Since $p^t R$ is in $\mathcal{V}$, we have $p^{n(p)}(p^t R) = 0$, so $n(p) + t \geq m(p)$, i.e.

$$n(p) \geq m(p) - t = \begin{cases} (m(p) - 1)/2 & \text{if } m(p) \text{ is odd} \\ m(p)/2 & \text{if } m(p) \text{ is even} \end{cases}$$

$$= [\, m(p)/2 \,] \, .$$

The final assertion of the theorem is clear from our remarks above. //

3. <u>Non-associative Possibilites</u>. We shall not pursue in detail the subject of S-closed varieties of non-associative rings, but merely make two observations. Firstly it is clear that any variety $\{ A \mid \mathcal{L} A = 0 \}$ of non-associative rings is an S-variety. Secondly, among the S-closed varieties of associative rings we have the class of boolean rings, this being the variety generated by K_2 and being extension-closed. We have noted elsewhere [2] that the variety of non-associative rings defined by the identity $x^2 = x$ is not extension-closed. We now present an example to show that this variety is not S-closed, either.

<u>Example 3.1</u>. Let A be an algebra over K_2 with basis $\{ u,v,w \}$ and multiplication table

	u	v	w
u	u	0	0
v	0	v	0
w	v	0	w

Then $\langle v,w \rangle \cong K_2 \oplus K_2 \cong \langle u,v \rangle$, so $\langle u,v \rangle$ and $\langle v,w \rangle$ satisfy $x^2 = x$; they are also ideals of A. But $\langle u,v \rangle$ +

$+ \langle u,w \rangle = \langle u,v,w \rangle = A$, while $(u + v + w)^2 = u^2 + v^2 + + wu + w^2 = u + v + v + w = u + w$, so A doesn't satisfy $x^2 = x$.

R e f e r e n c e s

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